Prenex Normal Form Theorem

— Exercises 28

Course:	701
Instructor:	Dr. Farmer
Presentation by:	Huidong(Dennis) Tian

Abstract

- What is Prenex Normal Form (PNF)
- Prove Prenex Normal Form Theorem
- Steps for transforming to PNF
- Example of transforming to PNF
- Motivation of transforming to PNF

What is Prenex Normal Form (PNF)

j is in prenex form (prenex normal form) if it is of the form

$$Q_1X_1\ldots Q_nX_n y$$

where each Q_i Î{$,"}, X_i are distinct predicate parameters, $Q_1X_1\ldots Q_nX_n$ are called quantified prefix, and y is quantifier free.

Prenex Normal Form Theorem

Every formula j has a logically equivalent formula j' in prenex form.

i.e. for every formula j of L (a language of first-order logic), there is a formula j' of L in prenex normal form such that j « j' is valid.

Proof by Induction

1. Base Case: $P(t_1,\dots,t_k)$
2. φ is $\neg \psi$:
3. φ is $\forall x.\ \psi$:
4. $\varphi = \varphi_1 \vee \varphi_2$:

Convention:

We denote φ' as PNF of φ.

2. φ is $\neg\psi$:

a) $\psi' = Q_1 x_1 \ldots Q_n x_n \theta$ (by induction)

where θ is quantifier free

b) $\neg\psi \Leftrightarrow \neg Q_1 x_1 \ldots Q_n x_n \theta \Leftrightarrow \underline{Q}_1 x_1 \ldots \underline{Q}_n x_n \neg\theta$

where $\underline{Q}_i$ is dual of Q_i, i.e. if $Q_i = \exists$ then $\underline{Q}_i = \forall$,

if $Q_i = \forall$ then $\underline{Q}_i = \exists$

notice: $\neg\forall x.\ \varphi \Leftrightarrow \exists x.\ \neg\varphi$

3. φ is $\forall x.\ \psi$:

a) $\psi' = Q_1 x_1 \ldots Q_n x_n \theta[x \rightarrow x']$ (by induction)

where x' is a variable that does not occur in ψ

b) $\forall x.\ \psi \Leftrightarrow \forall x' Q_1 x_1 \ldots Q_n x_n \theta[x \rightarrow x']$

notice: if x appears free in ψ, then we do not need to substitute x by x'

4. φ is $\varphi_1 \vee \varphi_2$:

a) Take φ_1 to φ_1' and take φ_2 to φ_2' (by induction)

notice: by using renaming to make quantified variables of φ_1' do not occur in φ_2' and vice versa.

b) Let $\varphi_1' = p_1\theta_1$ and $\varphi_2' = p_2\theta_2$

where **p_i** is the **quantified prefix** of a formula

c) $\varphi_1 \vee \varphi_2 \Leftrightarrow (p_1)(p_2)(\theta_1 \vee \theta_2)$

notice: we can shuffle **p_1** and **p_2**, i.e. the order of quantified variable in quantified prefix does not matter

Steps for transforming to PNF

1. Eliminate implication and equivalence constructor

   $p \Leftrightarrow q \equiv p \Rightarrow q \lor q \Rightarrow p$ 1.1

   $p \Rightarrow q \equiv \neg p \lor q$ 1.2

2. Move negation down to atomic formulae

   $\neg (p \lor q) \equiv \neg p \land \neg q$ 2.1

   $\neg (p \land q) \equiv \neg p \lor \neg q$ De Morgan law 2.2

   $\neg \forall x.\ \varphi \Leftrightarrow \exists x.\ \neg\varphi$ 2.3

   $\neg \exists x.\ \varphi \Leftrightarrow \forall x.\ \neg\varphi$ 2.4

Steps for transforming to PNF Con't.

3. Rename bound variables if necessary
4. Move quantifiers to the left

$\forall x\ \varphi(x) \vee \psi \equiv \forall x\ (\varphi(x) \vee \psi)$	4.1
$\forall x\ \varphi(x) \wedge \psi \equiv \forall x\ (\varphi(x) \wedge \psi)$	4.2
$\forall x\ \varphi(x) \wedge \forall x\ \psi(x) \equiv \forall x\ (\varphi(x) \wedge \psi(x))$	4.3
$\forall x\ \varphi(x) \vee \forall x\ \psi(x) \equiv \forall x \forall y\ (\varphi(x) \vee \psi(x)[x \rightarrow y'])$	4.4
$\exists x\ \varphi(x) \vee \exists x\ \psi(x) \equiv \exists x\ (\varphi(x) \vee \psi(x))$	4.5
$\exists x\ \varphi(x) \wedge \exists x\ \psi(x) \equiv \exists x \exists y\ (\varphi(x) \wedge \psi(x)[x \rightarrow y'])$	4.6

transforming to PNF Example

$\forall x_0 (\forall x_1 \varphi(x_1, x_0) \rightarrow \exists x_1 \psi(x_0, x_1))$

$\Leftrightarrow \forall x_0 (\neg \forall x_1 \varphi(x_1, x_0) \vee \exists x_1 \psi(x_0, x_1))$	1.2
$\Leftrightarrow \forall x_0 (\exists x_1 \neg \varphi(x_1, x_0) \vee \exists x_1 \psi(x_0, x_1))$	2.3
$\Leftrightarrow \forall x_0 (\exists x_1 \neg \varphi(x_1, x_0) \vee \exists x_2 \psi(x_0, x_2))$	3
$\Leftrightarrow \forall x_0 \exists x_1 \exists x_2 (\neg \varphi(x_1, x_0) \vee \psi(x_0, x_2))$	4.5
$\Leftrightarrow \forall x_0 \exists x_1 \exists x_2 (\varphi(x_1, x_0) \rightarrow \psi(x_0, x_2))$	reintroduce $\rightarrow$

Motivation

- It simplifies the surface structure of the sentence.
- It is useful for Skolemization
- It is useful for automated theorem proving

QUESTIONS