Presentation

Formalize Boolean Algebra in FOL

Topic 21 (Exercise 31 e)

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Boolean Algebra

- An abstract mathematical system primarily used in computer science and in expressing the relationships between sets
- (Paul 1963, p.8) A set together with
  - Two distinct and distinguished elements: 0, 1
  - Two 2-placed operations: +, *
  - A 1-placed operation: ¯
  - Satisfying several laws

- commutative laws
- associative laws
- distributive laws
- idempotent laws
- absorption laws
- de Morgan's laws
- laws of zero and one
- law of double negation

Formalize Boolean Algebra in FOL

- *Theory* $T=(L, \Gamma)$
- $L=(V, \{0,1\}, \{+,*,\bar{}\}, \emptyset)$
  - $L = (V, C, F, P)$
  - $C= \{0,1\}$ where 0, 1 are distinct distinguished constant symbols
  - $F =\{+,*,\bar{}\}$ where + , * are 2-placed function symbols, $\bar{}$ is 1-placed function symbol
  - $P = \emptyset$

Axioms

- Associative Laws
  - $x+(y+z)=(x+y)+z$ $\quad$ $x*(y*z)=(x*y)*z$
- Commutative Laws
  - $x+y=y+x$ $\quad$ $x*y=y*x$
- Idempotent Laws
  - $x+x=x$ $\quad$ $x*x=x$
- Distributive Laws
  - $x+(y*z)=(x+y)*(x+z)$ $\quad$ $x*(y+z)=(x*y)+(x*z)$
- Absorption Laws
  - $x+(x*y)=x$ $\quad$ $x*(x+y)=x$

- De Morgan laws
  - $\overline{x+y} = \overline{x} * \overline{y}$ $\quad\quad$ $\overline{x*y} = \overline{x} + \overline{y}$
- Laws of zero and one
  - $x+0=x$ $\quad\quad$ $x*0=0$
  - $x+1=1$ $\quad\quad$ $x*1=x$
  - $0 \neq 1$
  - $x+\overline{x}=1$ $\quad\quad$ $x*\overline{x}=0$

  $\overline{0}=1$; $\overline{1}=0$
- Law of Double negation
  - $\overline{\overline{x}}=x$

Example of Boolean Algebra

- $<D, \cup, \cap, ^{-}, \varnothing, X>$
  - $\varnothing$, *X are empty set and full set*
  - $\cup$, $\cap$, $^{-}$ *are union, intersection, complementation operator*
- $<D, \vee, \wedge, \neg, true, false>$

Partial Order $\leq$

- A Partial order can be defined on *D by* $x \leq y$ *iff* $x+y=y$.
- *Given any two elements* $x,y \in D$*,*
  - *Least upper bound of x and y is* $x+y$
  - *Greatest lower bound of x and y is* $x*y$
- *Boolean algebra has the structure of lattice*

Reference

- **Model theory** *by C.C.Chang and H.J.Keisley*
- **Lectures on Boolean Algebra** *by Paul R. Halmos*
- **Boolean algebra and its applications** *by J. Eldon Whitesitt*