Cantor’s Paradox

An Example of a Set-Theoretic Paradox Other than Russell’s Paradox

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References and Notes

- M Pakkan, V Akman, "Issues in Commonsense Set Theory", *Artificial Intelligence Review*, 8(4), 1994
- J T Miller, "An Historical Account of Set-Theoretic Antinomies Caused by the Axiom of Abstraction", http://www.u.arizona.edu/~miller/finalreport/finalreport.html
- R N Andersen, Set Theory and the Construction of Numbers, http://www.uwec.edu/Academic/Curric/andersrn/sets.htm

Naive vs. Axiomatic Set Theory

Naive Set Theory

Loosely
Defined
Concepts

- **Example: Cantor's Definition of a Set**

Naive vs. Axiomatic Set Theory

Axiomatic Set Theory

Rigorously Defined Concepts

- **All of Cantor's Theorems can be rigorously described in terms of 3 Axioms**

Cantor’s Axioms

- A1) **Extensionality**: two sets are identical if they have the same elements
- A2) **Abstraction**: for any given property there is a set whose members are just those entities having that property
- A3) **Choice**: If B is a set, all of whose elements are non-empty sets, (no two of which have any elements in common), then there is a set C which has precisely one element in common with each element of By

Cantor's Axioms

A2) <u>Abstraction</u>: for any given property there is a set whose members are just those entities having that property

$$\exists y \, \forall \mathbf{x} \, [\mathbf{x} \in \mathbf{y} \leftrightarrow \varphi(\mathbf{x})]$$

Set-Theoretical Paradoxes

$$\exists y\ \forall x\ [x \in y \leftrightarrow \varphi(x)]$$

The set of all things which have the property of:

- **not being members of themselves** → **Russell's Paradox**
- **being an ordinal** → **Burali-Forti's Paradox**
- **being a set** → **Cantor's Paradox**

Cantor's Paradox

Also known as ***"The Paradox of Cardinality"***

- **Let C be the universal set and P(C) be its power set**

- **Since P(C) is a set (the set of all subsets of C) it must be contained within C (because C contains all sets), so**

$$|P(C)| \leq |C|$$

- **However, by Cantor's Theorem**

$$|C| < |P(C)|$$

Contradiction

Cantor’s Theorem

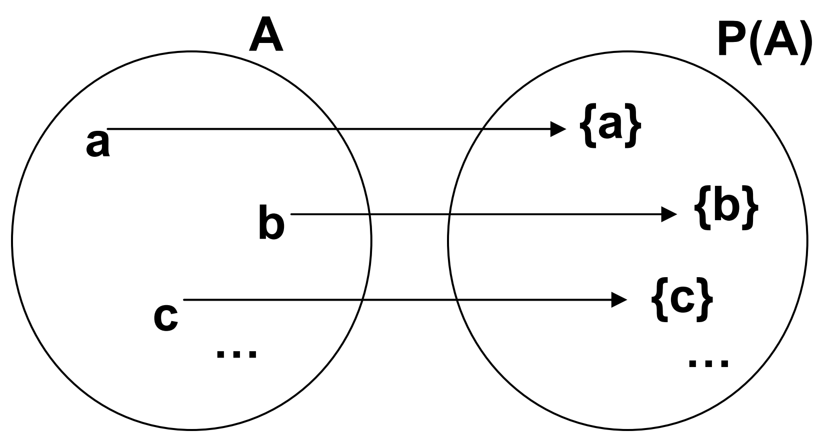
- **Let A be any set and P(A) be its power set**

$$|A| < |P(A)|$$

Cantor's Theorem…

- Recognize the existence of an *injective function*

$$f: A \rightarrow P(A)$$
$$a \mapsto \{a\}$$ *for instance*



Cantor's Theorem…

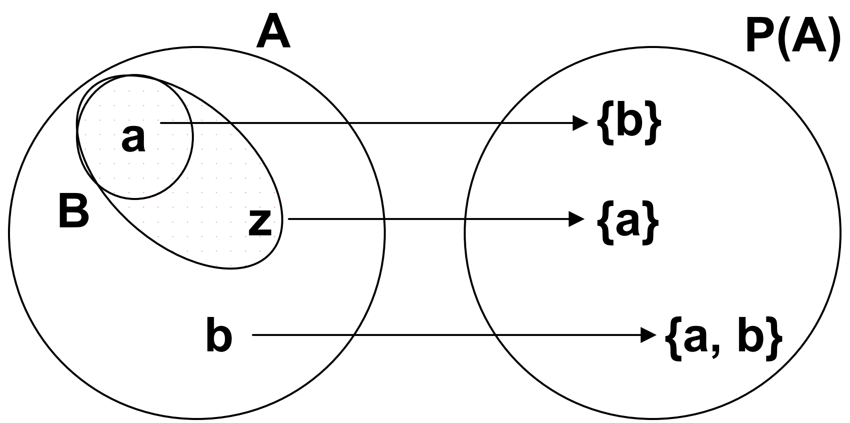
$\therefore \quad |A| \leq |P(A)|$

Now, if |A| = |P(A)|
then *f* would also have to be *surjective*

Cantor's Theorem…

Consider a function which maps as in the diagram

$f: A \rightarrow P(A)$, where f is surjective



$$B = \{x \in A \mid x \notin f(x)\} \subset A$$

$$\exists z \in A \mid f(z) = B$$

Then consider the consequences of a set defined as B

Cantor's Theorem

$z \in B \Rightarrow z \notin B$　　　　$z \notin B \Rightarrow z \in B$

Contradiction

So f cannot be surjective

$\therefore$ **f: A $\rightarrow$ P(A),**　　**Injective**

but not surjective

$$|A| < |P(A)|$$

Set Theoretical Axioms

- The Axiom of Abstraction basically states that a formula that specifies any property can define a set
- This non-constrained axiom allowed for the existence of some sets which do not exist
  - The barber's town
  - The Universal Set

Set Theoretical Axioms

- The Axioms of Zermello-Fraenkel Set Theory build in constraints to avoid the allowance of such sets.
  - Axiom of Foundation (A6)
  - Axiom of Separation (A7)
  - Axiom of Replacement (A8)

Questions??

Contents of today's presentation can be found at:

http://www.cas.mcmaster.ca/~wmfarmer/CAS- …
… 701- 02/contributions/index.html