# Compactness Theorem for First-Order Logic

CAS 701 Presentation
Exercise 30 part a

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## Compactness Theorem

Let □ be any set of formulas of first-order logic. Then □ is satisfiable if every finite subset of □ is satisfiable.

## Proof (Using Completeness Theorem)

- If □ is finite, then the proof is obvious since □ is a finite subset of itself so it's satisfiable.
- Proof by contradiction. Suppose □ is unsatisfiable.
- By Godel's completeness theorem, □ is inconsistent in F (sound and complete formal system for first-order logic)

$$\square|\square_F\ \square,\ \square\square$$

- There is a proof in F from□ for both □ and □□

- A proof consists of a <u>finite sequence</u> of formulas.
- Let P = set of formulas from ∏ which are used in the proof
- P is finite and is a subset of ∏
- P |□$_F$ □, □□
- P |= □, □□ since F is sound

This means that P is unsatisfiable. Contradicting the assumption, so ∏ is satisfiable.

## Question Overview

- Use the compactness theorem to prove that every first-order theory that has arbitrarily large finite models, has an infinite model.

- Let T = (L, ∏)

- Define

$$f\_k = \exists x_1 \exists x_2 \ldots \exists x_k .(x_1 \neq x_2) \wedge (x_1 \neq x_3) \wedge \ldots (x_{k-1} \neq x_k)$$

$$S = \Sigma \cup \{f\_k \mid k \geq 2\}$$ includes infinite number of formulas

- Any model satisfying S will be infinite.
- Let A be a finite subset of S and K be the maximum index of f_k in A.
- Every model with cardinality greater than or equal to K that satisfies $\Sigma$ will satisfy A.
- Every finite subset of S is satisfiable
- By using compactness theorem, we conclude that S is satisfiable and it's model should be infinite.
- Every model that satisfies S, will also satisfy $\Sigma$ so T has an infinite model.

## References:


1. http://cheng.ececs.uc.edu/cs543/4-29.html

2. http://www.trentu.ca/academic/math/sb/pcml/pcml-i-15.pdf