The Theory of a Complete Ordered Field

Presentation prepared and presented by Joe Fakhri
Date Revised: November 26, 2002

Outline

- Simple Type Theory
- Definitions
- Axioms of a Field
- Axioms of an Ordered Field
- Completeness Axiom
- Conclusion

Simple Type Theory

A theory of STT is the following tuple: T = (L, Γ)

A language of STT is a tuple $L = (\nu, \mathcal{C}, \tau)$ where:

- ν is an infinite set of symbols called variables
- $\mathcal{C}$ is set of symbols called constants
- τ: a total function $\mathcal{C} \rightarrow \mathcal{T}$

Γ : Axioms of T

Definitions

- The theory of a complete ordered field:

$(D, +, -, 0, \times, {}^{-1}, 1, <)$. ι is type of elements in D.

$\tau(+) = (\iota \to (\iota \to \iota))$	$\mathcal{C} = \{+, \times, {}^{-1}, -, <, 0, 1\}$
$\tau(-) = (\iota \to \iota)$	Notations:
$\tau(\times) = (\iota \to (\iota \to \iota))$	$a + b$ denotes $+(a, b)$
$\tau({}^{-1}) = (\iota \to \iota)$	$a + (-a)$ denotes $+(a, -(a))$
$\tau(0) = \iota$	$a \times b$ denotes $\times(a, b)$
$\tau(1) = \iota$	
$\tau(<) = (\iota \to (\iota \to *))$	

Field Axioms

- A field is a nonempty set with 2 functions × & + satisfying the following axioms:
  - Axiom 1: (Associative Laws)
    - $\forall a, b, c: \iota.\ (a + b) + c = a + (b + c)$
    - $\forall a, b, c: \iota.\ (a \times b) \times c = a \times (b \times c)$
  - Axiom 2: (Commutative Laws)
    - $\forall a, b: \iota.\ a + b = b + a$
    - $\forall a, b: \iota.\ a \times b = b \times a$

Axioms of a Field (continued)

- Axiom 3: (Distributive Law)
  - ∀a, b, c: ι. a × (b + c) = (a × b) + (a × c)
  - ∀a, b, c: ι. (a + b) × c = (a × c) + (b × c)
- Axiom 4: (Existence of identities)
  - ∀a: ι. a + 0 = a
  - ∀a: ι. a × 1 =a
- Axiom 5: (Existence of inverse)
  - Additive inverse: ∀a: ι. a + (−a) = 0
  - Multiplicative Inverse: ∀a: ι. ¬ (a=0) => a × ($_{-1}$(a)) = 1

Axioms of a Field (continued)

- Example of a field: the set of rational numbers **Q** and set of real numbers **R**.
- Additive and multiplicative identities of a field F are unique.
  - Proof: suppose e1 and e2 are both mult. identities in D then e1 = e1 $\times$ e2 and e2 = e2 $\times$ e1=> e1 = e2
  - Suppose e1 and e2 are both additive identities in F then e1=e1 + e2 and e2 = e2 + e1 => e1 = e2

Axioms of an Ordered Field

- An ordered field is a linearly ordered field by $<$
  - Axioms for the theory of linear orders is as follows:
    - Axiom 6: Irreflexivity: $\forall a: \iota.\ \neg(a < a)$
    - Axiom 7: Transitivity: $\forall a, b, c: \iota.\ (a < b) \wedge (b < c) \Rightarrow (a < c)$
    - Axiom 8: Trichotomy: $\forall a, b: \iota.\ (a < b) \vee (b < a) \vee (b = a)$
  - Just as the distributive law links together + and ×, we need rules to tell us how + and × affect the ordering of elements of our field, therefore we define the following 2 axioms:
    - Axiom 9: $\forall a, b, c: \iota.\ (a < b) \Rightarrow (a + c) < (b + c)$
    - Axiom 10: $\forall a, b: \iota.\ (0 < a) \wedge (0 < b) \Rightarrow (0 < a \times b)$

Axioms of an Ordered Field (continued)

- Example : we need to derive that: $(0 < a) \wedge (0 < b) \Rightarrow (0 < a + b)$

1. $(0 < a) \wedge (0 < b)$
2. $(0 + (-b)) < (b + (-b))$ (by axiom 9)
3. $-b < 0$ (from 2)
4. $(-b < 0) \wedge (0 < a) \Rightarrow (-b < a)$ (by axiom 8)
5. $(-b + b) < (a + b)$ (by axiom 9)
6. $0 < (a + b)$ (from 5) => derived

Axioms of an Ordered Field (continued)

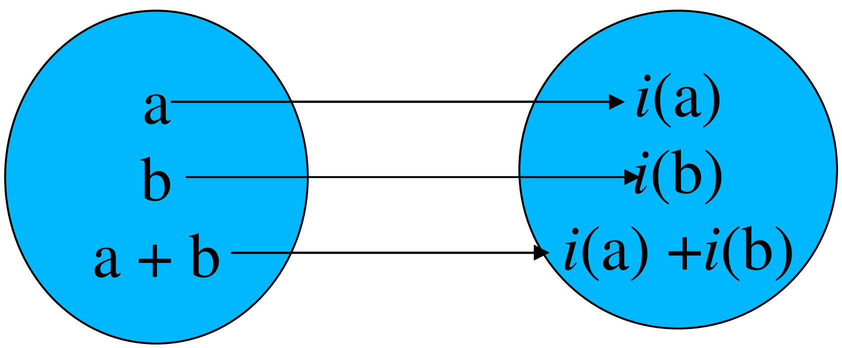
- Examples:
  - The Rational numbers **Q** are an ordered field
  - Real numbers **R** are an ordered field
  - Let F be the set $\mathbf{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$ on which + and × are defined. F is a field under these operations but when we define an ordering $0<1<2<3<4<5<6$ on **F**, then **F** is not an ordered field under this ordering.
    - Solution: with this given ordering we have $0 < 1$ but if we add 6 then we produce a contradiction. $0+6 < 1 + 6 \Rightarrow 6 < 0$

Completeness Axiom

- Every non-empty set which is bounded above has a least upper bound.
- Definition of bounded above: A subset **S** of an ordered field **F** is bounded above, if there exist M∈**F** such that x< M for all x ∈ **S**
- Completeness Axiom: (Axiom 11)
  - $\forall S: (\iota \rightarrow *). ((\exists x:\iota. S(x)) \Rightarrow [\exists b: \iota. (upper (S, b) \wedge \forall c: \iota. upper (S, c) \Rightarrow b < c)])$
  - Where: $upper (S, a) = \forall x: \iota. S(x) \Rightarrow x < a$

Completeness Axiom (continued)

- The existence of a complete ordered field is the set of real numbers denoted by **R**
- If **F**1 and **F**2 are both complete ordered fields then we can show that there exist a bijective function i: **F**1 $\rightarrow$ **F**2 such that $i(a+b) = i(a) + i(b)$, $i(a \times b) = i(a) \times i(b)$, and $a < b \Leftrightarrow i(a) < i(b)$



Therefore **F**1 and **F**2 are the same. The function i is called an order isomorphism

Conclusion

- The existence of such an order isomorphism shows that the real numbers are essentially unique.

  Therefore the real numbers are the only

  Complete Ordered Field

References

- http://www.math.unl.edu/~webnotes/classes/class03/class03.htm
- http://www.gap-system.org/~john/analysis/Tutorials/T3.html
- http://www.math.louisville.edu/~lee/RealAnalysis/ra_sect02.pdf
- http://www.math.niu.edu/~rusin/known-math/index/12-XX.html
- http://www.math.louisville.edu/~lee/02Spring501/chapter2.pdf
- http://mathworld.wolfram.com/Field.html

Note: This presentation will be posted on:
http://www.cas.mcmaster.ca/~fakhrijm/home.html