Primitive Recursive Definition of the Exponential Function on N

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Primitive Recursive

- Take n natural numbers as inputs, produce a natural number
- Basic primitive functions are given by three axioms

Axioms

- The “Constant Function” 0
  $Z = \lambda x.0$

- The “Successor Function”, as defined in Peano arithmetic
  $S = \lambda x.(x+1)$

- The “Projection Function” P_i^n, which takes n arguments and returns the $i^{th.}$
  $P_i^n = \lambda x_1 \cdots x_n . x_i$

Exponential Function

- The exponential function computes x^y
- x^y is simply x multiplied by itself y times
- $x \cdot x$ is simply x added to itself x times
- $x+x$ is simply $x+1$ or $S(x)$ x times

Recursive Definition of Addition

- Sum function f = λx,y.(x + y)

$$\begin{cases} f(x,0) & = x \\ f(x,y+1) & = f(x,y)+1 \end{cases}$$

- In recursive notation

$$\begin{cases} f(x,0) & = g(x) \\ f(x,y+1) & = h(x,y,f(x,y)) \end{cases}$$

Addition cont'd

- g, h $\in$ PR, let g(x) = x, and h(x,y,z) = z + 1.

$$g(x) = P_1^1(x), \text{and } h(x, y, z) = S(P_3^3(x, y, z))$$

- a PR derivation of f is:

$$f_1 \leftarrow P_1^1$$

$$f_2 \leftarrow S$$

$$f_3 \leftarrow P_3^3$$

$$f_4 \leftarrow f_2 \circ f_3$$

$$f_5 \leftarrow f_1 \circ f_4$$

Recursive Definition of Multiplication

- Prod function f = λx,y.(x $\cdot$ y)

$$\begin{cases} f(x,0) &= 0 \\ f(x, y+1) &= f(x, y) + x \end{cases}$$

- In recursive notation

$$\begin{cases} f(x,0) &= g(x) \\ f(x, y+1) &= h(x, y, f(x, y)) \end{cases}$$

Multiplication cont'd

- g, h $\in$ PR

$$g(x) = Z(x), \text{and}$$

$$h(x, y, z) = z + x$$

$$= sum(z, x)$$

$$= sum(P_3^3(x, y, z), P_1^3(x, y, z))$$

- a PR derivation of f is:

$$f_6 \leftarrow Z$$

$$f_7 \leftarrow P_3^3$$

$$f_8 \leftarrow P_1^3$$

$$f_9 \leftarrow f_5 \circ f_7 \circ f_8$$

$$f_{10} \leftarrow f_6 \circ f_9$$

Recursive Definition of the Exponential Function

- Exp function f = λx,y.(x ^ y)

$$\begin{cases} f(x,0) &= 1 \\ f(x,y+1) &= f(x,y)\cdot x \end{cases}$$

- In recursive notation

$$\begin{cases} f(x,0) &= g(x) \\ f(x,y+1) &= h(x,y,f(x,y)) \end{cases}$$

Exponential Function cont'd

- g, h $\in$ PR

$$g(x) = S(Z(x)), \text{and}$$

$$h(x, y, z) = z \cdot x$$

$$= prod(z, x)$$

$$= prod(P_3^3(x, y, z), P_1^3(x, y, z))$$

- a PR derivation of f is:

$$f_{11} \leftarrow S(Z)$$

$$f_{12} \leftarrow P_1^3$$

$$f_{13} \leftarrow P_3^3$$

$$f_{14} \leftarrow f_{10} \circ f_{12} \circ f_{13}$$

$$f_{15} \leftarrow f_{11} \circ f_{14}$$

Conclusion

Primitive recursion allows us to reduce complicated mathematical operations such as the exponential function to very simple operations, well defined by Peano arithmetic.

References

- Dr. Zucker: Recursive Function Theory
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  http://www-users.cs.york.ac.uk/~susan/cyc/p/primrec.htm