Exercise 9

Let f : A ® B and g : B ® C be total, and let h = $g \circ f$: A ® C be the composition of g and f.

(a) Prove that, if f and g are injective, then h is injective, but the converse is false.

(b) Prove that, if f and g are surjective, then h is surjective, but the converse is false.

By

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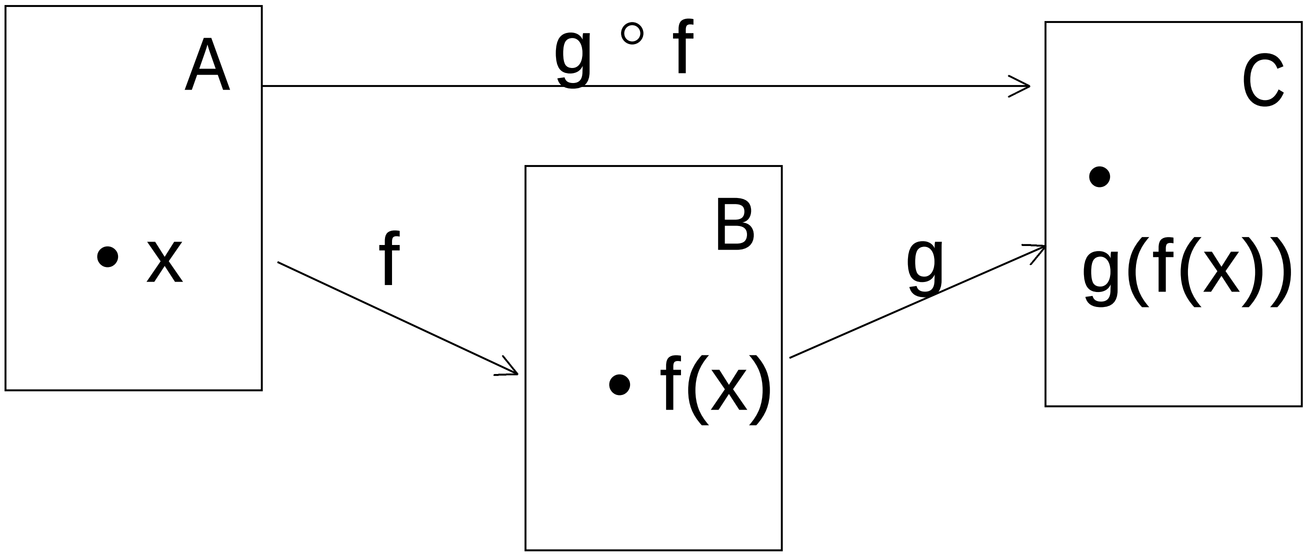
The definition of composition

The composition of the functions $g \circ f$ is defined by the assignment

$$\forall x \in D_{g \circ f},\ g \circ f(x) = g(f(x)).$$



f, g are injective $\Rightarrow$ h is injective

Proof:

Suppose $h(a_1) = h(a_2)$, where $a_1, a_2 \in$ A.

Then, by the definition of composition,

$$g(f(a_1) = g(f(a_2)).$$

Since g is injective, we know $f(a_1) = f(a_2)$.

Since f is injective, we know $a_1 = a_2$.

Thus, $h(a_1) = h(a_2) \Rightarrow a_1 = a_2$, which is the definition of an injective function. Thus, h is injective.

f, g are surjective $\Rightarrow$ h is surjective

Proof:

Let c be an arbitrary element of C.

Since g is surjective, $\exists\, b \in$ B such that $g(b) = c$. Furthermore, since f is surjective, $\exists\, a \in$ A such that $f(a) = b$.

Therefore,

$$g(f(a)) = g(b) = c, \text{ or } g \circ f(a) = c.$$

h is surjective.

h is injective ⇒ g is injective ?
h is surjective ⇒ f is surjective ?

Counterexample:

$f : \mathrm{A} \rightarrow \mathrm{B}$ and $g : \mathrm{B} \rightarrow \mathrm{C}$;
A={a}, B={b, d}, C={c},

$f(a) = b$,

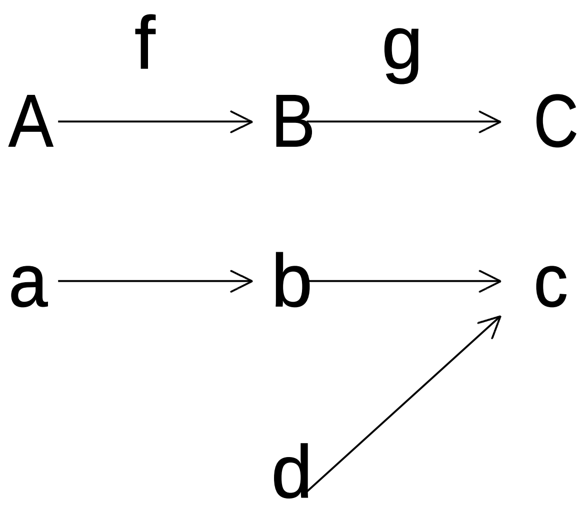
$g(b) = c$, $g(d) = c$,

$h(a) = g \circ f(a) = c$.

h is surjective and injective,

but f is not surjective,

and g is not injective.



h is injective $\Rightarrow$ f is injective

Proof:

Suppose $f(a_1) = f(a_2)$, $f(a_1), f(a_2) \in$ B.

Because g is total, with the definition of composition, we have

$$h(a_1) = g(f(a_1)) = g(f(a_2)) = h(a_2).$$

Thus the injectivity of h provides $a_1 = a_2$ as desired.

h is surjective $\Rightarrow$ g is surjective

Proof:

Let c be an arbitrary element of C. Since h is surjective, there must $\exists\, a \in$ A such that $h(a) = c$.

By the definition of composition, $\exists\, b \in$ B, $f(a)=b$ and $g(b) = c$.

Thus, g is surjective.

Conclusion

- if f and g are injective, then h is injective.
- if f and g are surjective, then h is surjective.
- if f and g are bijective, then h is bijective.

- if $h = g \circ f$ is injective, then f is injective.
- if $h = g \circ f$ is surjective, then g is surjective.
- if $h = g \circ f$ is bijective, then f is injective, g is surjective.