Theory of Lattices In FOL

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What is Lattices (view of set theory)

Lattice $(D, \leq, \cup, \cap)$

$\leq \subseteq D \times D$, $\cup: D \times D \rightarrow D$, $\cap: D \times D \rightarrow D$

- $(D, \leq)$

  Partial order set

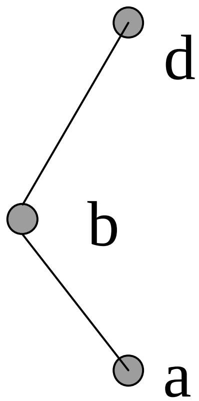
- $\cup$ (Join)

  Every subset {a, b} has a (unique) least upper bound in D, denote as $a \cup b$, a join b

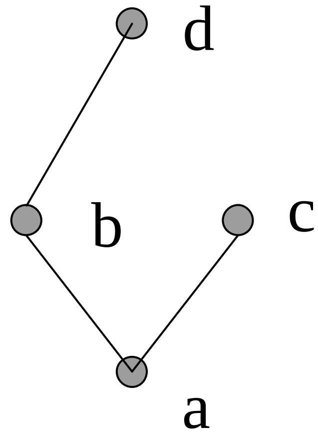
- $\cap$ (meet)

  Every subset {a, b} has a (unique) greatest lower bound in D, denote as $a \cap b$, a meet b

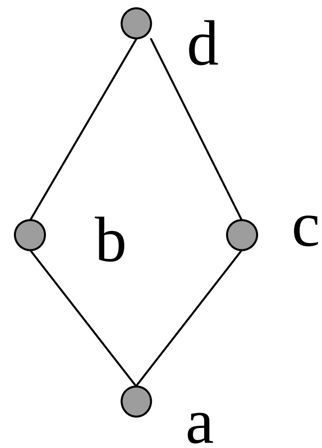
Lattices and Not Lattices



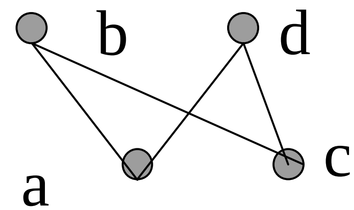
(1) Y



(2) N



(3) Y



(4) N

1. total order is lattice $x \leq y \Rightarrow (x \cup y) = y$ (Least upper bound)

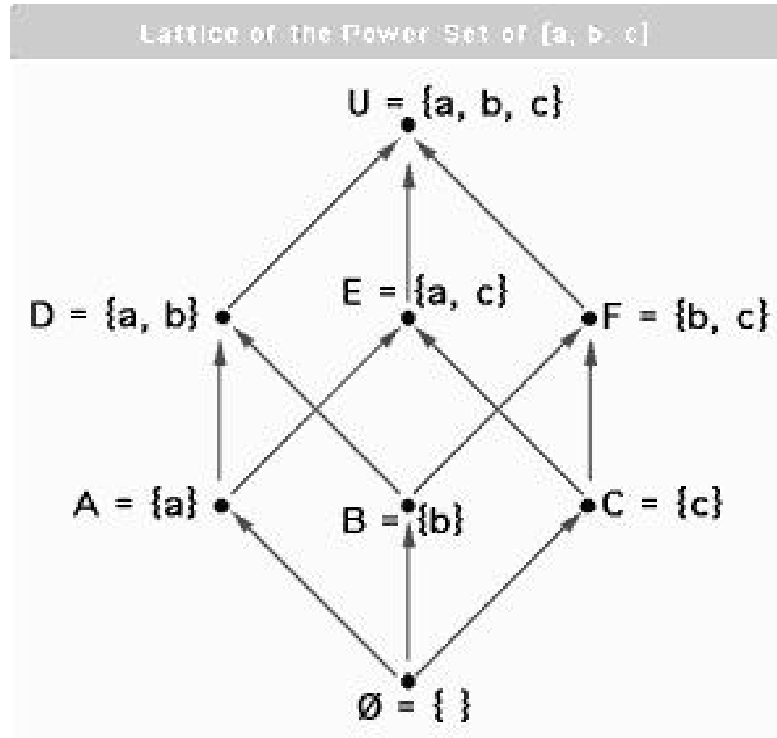
   $(x \cap y) = x$ (Greatest lower bound)

2. form lattice from partial order.

Example of Lattices

Lattice (D, ≤, ∪, ∩)

D: Power set of {a,b,c}
≤:   if x ≤ y, iff x ⊆ y
∪ (join) : union of subsets of power set
∩ (meet): intersection of subset sets of power set



Theory of Lattices

$T = \{ L, \Gamma \}$

$L = \{V, \Phi, \{\cup, \cap\}, \{\leq\}\}$

Γ: Axioms

1. $\forall x.\ x \leq x$ *(Reflexive)*
2. $\forall x, y.\ (x \leq y) \wedge (y \leq x) \Rightarrow (x = y)$ *(Antisymmetry)*
3. $\forall x, y, z.\ (x \leq y) \wedge (y \leq z) \Rightarrow (x \leq z)$ *(Transitivity)*
4. $\forall x, y, z\ \exists u.\ (((x \leq u) \wedge (y \leq u)) \wedge ((x \leq z) \wedge (y \leq z) \Rightarrow (u \leq z)))$

   (Least upper bound)
5. $\forall x, y, z\ \exists u.\ (((u \leq x) \wedge (u \leq y)) \wedge ((z \leq x) \wedge (z \leq y) \Rightarrow (z \leq u)))$

   (Greatest lower bound)

What is Lattices (view of algebra)

Lattice $(D, \cup, \cap)$

$\cup: D\times D \rightarrow D$, $\cap: D\times D \rightarrow D$

- $x \cap x = x$, $x \cup x = x$ *(Idempotency)*
- $x \cap y = y \cap x$, $x \cup y = y \cup x$ *(Commutativity)*
- $(x \cap y) \cap z = x \cap (y \cap z)$ *(Associativity)*

  $(x \cup y) \cup z = x \cup (y \cup z)$
- $x \cap (x \cup y) = x$, $x \cup (x \cap y) = x$ *(Absorption identities)*

Theory of Lattices (view of algebra)

$T = \{ L, \Gamma \}$

$L = \{V, \Phi, \{\cup, \cap\}, \Phi\}$

Γ: Axioms

- $\forall x.\ x \cap x = x,\ x \cup x = x$ *(Idempotency)*
- $\forall x, y.\ x \cap y = y \cap x$ *(Commutativity)*
  $\forall x, y.\ x \cup y = y \cup x$
- $\forall x, y, z.\ (x \cap y) \cap z = x \cap (y \cap z)$ *(Associativity)*
  $\forall x, y, z.\ (x \cup y) \cup z = x \cup (y \cup z)$
- $\forall x, y.\ x \cap (x \cup y) = x$ *(Absorption identities)*
  $\forall x, y.\ x \cup (x \cap y) = x$

Lattices and Boolean Algebra

Boolean lattice

- Boolean algebra is a boolean lattice
- Is a lattice a boolean algebra?

Complementarity

$x \cup y = 1,\ x \cap y = 0$

1: universal upper bound   0: universal lower bound

Distributivity

$x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$

$x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$

Conclusion

- A lattice as an algebra and a lattice as a poset are "equivalent" concepts.

  – 1. Let the poset L = (D, ≤, ∪, ∩) be a lattice. Set x ∩ y = GLB {x, y} and x ∪ y = LUB {x, y} . Then the algebra L = (D, ∪, ∩) is a lattice.

  – 2. Let the algebra L = (D, ∪, ∩) be a lattice. Set x ≤ y iff x ∩ y = x . Then L = (D, ≤, ∪, ∩) is a poset, and the poset is a lattice.

References

1. Introduction to Lattice theory *by D.E. Rutherford*
2. Lattice theory *by George Gratzer*
3. Lattice theory *by Thomas Donnellan*
4. http://www.rci.rutgers.edu/~cfs/472_html/Learn/SimpleLattice_472.html