Natural Deduction System

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Natural Deduction

- People seem to use a method for constructing informal arguments or proofs in natural language.
- Informal proofs exhibit `patterns of reasoning' such as the following:

if Logic is fun, then Stephen is happy Logic is fun

Stephen is happy

This instance of Modus Ponens seems quite natural.

Question: Can we formulate a system of deduction based entirely on `natural laws'?

Natural Deduction

In 1935, the German mathematician Gerhard Gentzen introduced natural deduction system in his paper. In this paper, Gentzen set down clearly his idea: I intend to set up a formal system which comes as close as possible to actual reasoning.

The formal system of Natural Deduction consists of the following components:

- 1. The language of Propositional Logic
- 2. Various rules of inference:
 - Introduction rules:
 produce complex statements from smaller statements
 by introducing connectives
 - Elimination rules: produce simpler statements from complex statements by eliminating connectives.

Note: Natural Deduction system has no axioms.

Introduction Rules

■ Conjunction Introduction (∧ I):

$$\frac{\mathsf{A} \quad \mathsf{B}}{\mathsf{A} \land \mathsf{B}}$$

■ Disjunction Introduction (∨ I):

$$\frac{A}{A \lor B}$$
 $\frac{B}{A \lor B}$

■ Implication Introduction (\rightarrow I): if B can be derived from assumption A, then we can get conclusion A \rightarrow B

Elimination Rules

■ Conjunction Elimination (∧ E):

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

■ **Disjunction Elimination** (∨ E): if C can be derived from A∨B, then A and B are crossed out. We get conclusion C.

■ Implication Elimination (→E):

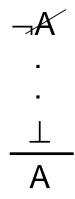
$$\frac{A \quad A \rightarrow B}{B}$$

Proof By Contradiction

Consider the following method of reasoning:

To prove that some statement A holds. Assume that $\neg A$ holds. If this assumption leads to a contradiction, then we can conclude that $\neg A$ cannot hold. so A must hold.

This proof method is known as Proof by Contradiction, or reductio ad absurdum (RAA).



⊥: the constant value false

Other rules

 \blacksquare \bot : from falsum, we can get any conclusion C

$$\frac{\perp}{C}$$

■ Id (identity): any formula can be deduced from itself

$$\frac{A}{A}$$

Note: $\neg A$ is defined as $A \rightarrow \bot$

Example

We show that $|-(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$ the set of assumption is empty

The proof proceeds as follows:

In the proof proceeds, all assumptions should be discharged.

Summary

- Natural deduction was presented as abstract system of propositional logic based on rules of derivation which are intended to capture ordinary human reasoning.
- In natural deduction system, there are no axioms, but many rules of inference including a formalization of proof by contradiction. The inference rules fall into two groups: Introduction Rules and Elimination Rules.
- Most proofs involve the making and discharging of assumptions.