Natural Deduction System

By Zhuomei Wu

Natural Deduction

- People seem to use a method for constructing informal arguments or proofs in natural language.
- Informal proofs exhibit `patterns of reasoning‘ such as the following:

  if Logic is fun, then Stephen is happy
  Logic is fun

  Stephen is happy

  This instance of Modus Ponens seems quite natural.

Question: Can we formulate a system of deduction based entirely on `natural laws'?

Natural Deduction

In 1935, the German mathematician Gerhard Gentzen introduced natural deduction system in his paper. In this paper, Gentzen set down clearly his idea: <u>I intend to set up a formal system which comes as close as possible to actual reasoning.</u>

The formal system of Natural Deduction consists of the following components:

1. The language of Propositional Logic
2. Various rules of inference:
   - **Introduction rules:**
     produce complex statements from smaller statements by introducing connectives
   - **Elimination rules:**
     produce simpler statements from complex statements by eliminating connectives.

Note: Natural Deduction system has no axioms.

Introduction Rules

- **Conjunction Introduction** ($\wedge$ I) :

$$\frac{A \qquad B}{A \wedge B}$$

- **Disjunction Introduction** ($\vee$ I):

$$\frac{A}{A \vee B} \qquad\qquad \frac{B}{A \vee B}$$

- **Implication Introduction** ($\rightarrow$I): if B can be derived from assumption A, then we can get conclusion A $\rightarrow$B

$$\frac{\begin{array}{c}\cancel{A}\\ \vdots \\ B\end{array}}{A \rightarrow B}$$

Elimination Rules

- **Conjunction Elimination** ($\wedge$ E) :

$$\frac{A \wedge B}{A} \qquad\qquad \frac{A \wedge B}{B}$$

- **Disjunction Elimination** ($\vee$ E): if C can be derived from A$\vee$B, then A and B are crossed out. We get conclusion C.

$$\frac{A \vee B \quad \begin{array}{c}\cancel{A}\\ .\\ .\\ C\end{array} \qquad\qquad \begin{array}{c}\cancel{B}\\ .\\ .\\ C\end{array}}{C}$$

- **Implication Elimination** ($\to$E):

$$\frac{A \quad A \to B}{B}$$

Proof By Contradiction

Consider the following method of reasoning:

To prove that some statement A holds. Assume that ¬A holds. If this assumption leads to a contradiction, then we can conclude that ¬A cannot hold. so A must hold.

This proof method is known as Proof by Contradiction, or reductio ad absurdum (RAA).

$$\frac{\begin{array}{c}\cancel{\neg A}\\ . \\ . \\ \bot\end{array}}{A}$$

⊥: the constant value false

Other rules

- ⊥ : from falsum, we can get any conclusion C

$$\frac{\bot}{C}$$

- Id (identity) : any formula can be deduced from itself

$$\frac{A}{A}$$

Note: $\neg A$ is defined as $A \to \bot$

Example

We show that |- (¬B→¬ A)→(A→B)

the set of assumption is empty

The proof proceeds as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{(1)}{\cancel{\neg B}} \quad \overset{(2)}{\cancel{\neg B \to \neg A}}}{\neg A} \qquad \dfrac{\overset{(3)}{\cancel{A}}}{\neg A \to \bot}}{\bot}}{B}}{A \to B}}{(\neg B \to \neg A) \to (A \to B)}$$

Assumption

→E, by definition of ¬A

→E

RAA,using (1)

→I, using (3)

→I, using (2)

In the proof proceeds, all assumptions should be discharged.

Summary

- Natural deduction was presented as abstract system of propositional logic based on rules of derivation which are intended to capture ordinary human reasoning.

- In natural deduction system, there are no axioms, but many rules of inference including a formalization of proof by contradiction. The inference rules fall into two groups: Introduction Rules and Elimination Rules.

- Most proofs involve the making and discharging of assumptions.