$f(A) = \{y \mid y=f(x) \wedge x \in A\}$

If A and B are infinite:
We try to partition A into $A_1$ and $A-A_1$, such that

$h(x)=f(x)$ if $x \in A_1$
$h(x)=g^{-1}(x)$ o/w

To have h bijective, we need:

$$A-A_1=g(B-f(A_1)) \quad [*]$$

[redacted]

To find $A_1$ we first consider all candidates $A_0$s satisfies:

$A-g(B)\subseteq A_0$ [&]

$(g^\circ f)(A_0)\subseteq A_0$ [&&]

Do we actually have any $A_0$ at all?

Yes.

Set A satisfies [&] and [&&]. But the two equations are still far from [*].

Now we name AA to be a set of sets composed of all qualified $A_0$s and let set $A_2$ be the intersection of all the elements of AA. ($A_2 = \cap A_0$) Obviously, $A_2$ is a subset of all $A_0$s in AA.

$A\text{-}g(B) \subseteq A_0 \implies A\text{-}g(B) \subseteq \cap A_0 = A_2$

$(g \circ f)(A_2) \subseteq (g \circ f)(A_0) \implies (g \circ f)(A_2) \subseteq \cap A_0 = A_2$

So $A_2$ satisfies [&] and [&&] and therefore $A_2 \in AA$.

Now we prove that $A_2$ satisfies [**]:

$A-g(B)\subseteq A_2$

$(g\circ f)(A_2) \subseteq A_2 \implies (A-g(B))\cup(g\circ f)(A_2) \subseteq A_2$ [#]

- ⇨ $(g\circ f)\,((A-g(B))\cup(g\circ f)(A_2)) \subseteq (g\circ f)\,(A_2) \subseteq (A-g(B))\cup(A_2)$
- ⇨ $[(A-g(B))\cup(g\circ f)(A_2)\,] \in AA$.
- ⇨ $A_2 \subseteq (A-g(B))\cup(g\circ f)(A_2)$ [##]

From [#] and [##] we get $(A-g(B))\cup(g\circ f)(A_2) = A_2$ [###]

Next we prove that $g(B-f(A_2))$ and $A_2$ are disjoint:

[redacted]

If there exist any intersection between $g(B-f(A_2))$ and $A_2$ The elements in the intersection will have two pre-image from B, which is contradict to the assumption g is injective. So $g(B-f(A_2)) \subseteq A-A_2$

Now we can see $A - A_2 = A - g(B - f(A_2))$ and this is exactly [*] our $A_2$ is $A_1$ we talked before.

Conclusion:

Schroder Bernstein Theorem is useful because in many cases injection is easier to prove than surjection. So this theorem provide a new approach to prove a bijective functions or equinumerous sets.