Semantic Tableau for Propositional Logic

Fatima Issawi

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Presentation Outline

- Background
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- Why is it useful?
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- Soundness and Completeness
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Background

- Semantic tableau is a refutation based system:
  - To prove a formula x, start with the negation ¬x, and produce a contradiction.
  - Expand the formulas and the premises by tableau expansion rules based on the structure of the compound formulas, forming a tree.
  - Each branch represents a way the conjunction of the formulas at the root can be satisfied.

Background (2)

- A semantic tableau is a tree. Each branch of the tree represents a way the negation of the conclusion is true.
- If all branches lead to contradictions, then there is no way the formula could be true.

What does it mean?

- A branch is closed if a and ¬a both appear on the path from the root of the tree to the leaf.
- If all branches are closed, then the tableau is closed, and we can conclude the formula at the root is not satisfiable. Therefore, the negation is a tautology.
- So, to show that $\Phi \vDash \varphi$, it suffices to show that $\Phi \wedge \neg\varphi$ is unsatisfiable.

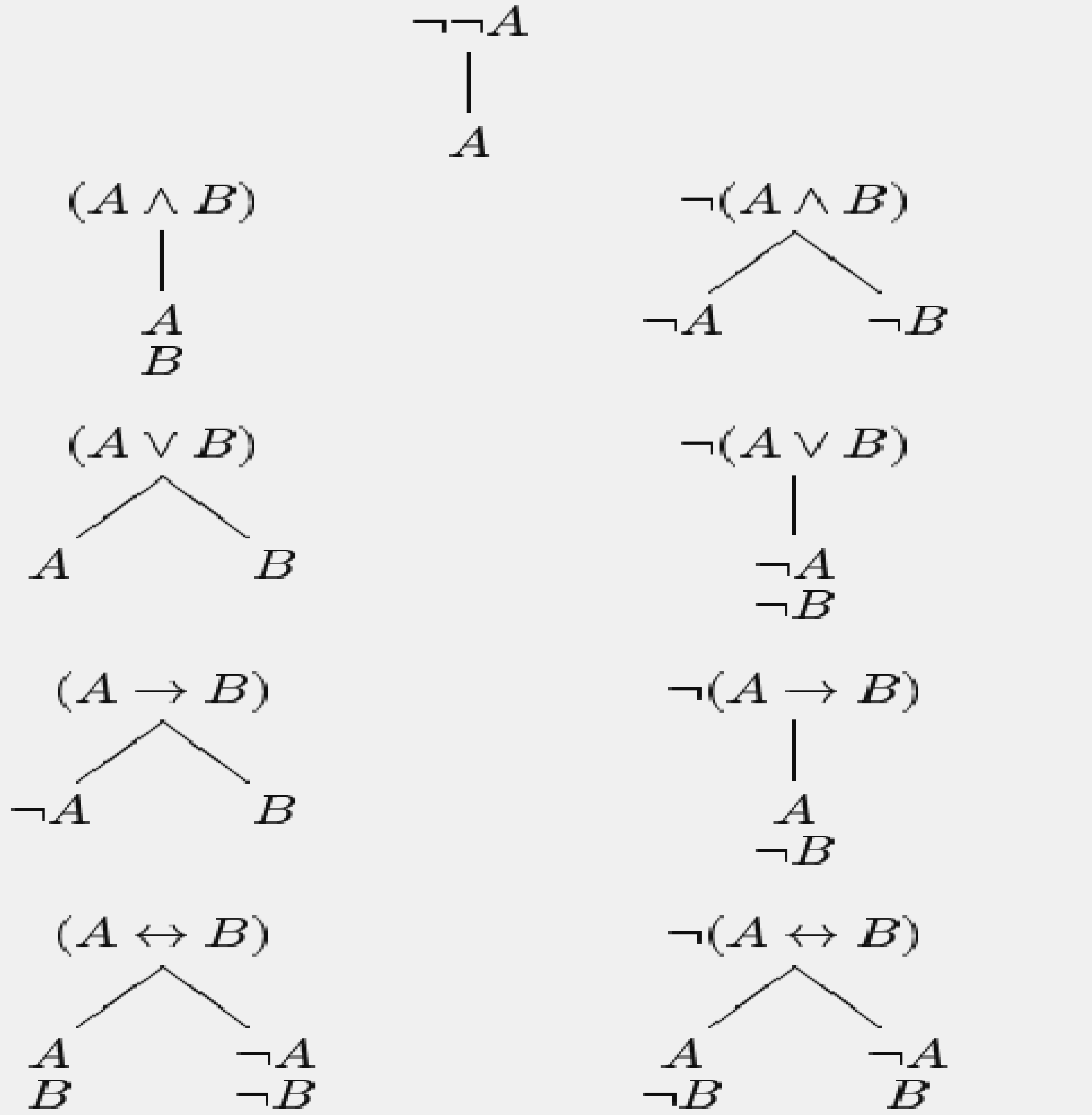
Why is this useful?

- Semantic tableau conducts a direct search for models.
- Naïve approaches, such as constructing a truth table, can take 2^n steps in the worst case for n propositional letters.
- Can be easily used to automate logical deduction efficiently.

Testing Entailment

- To test whether $\Phi \models \varphi$:
  - Form the set $\Phi \cup \{\neg \varphi\}$, and
  - Use Semantic Tableau to determine if the set is inconsistent (entailment holds), or consistent (entailment does not hold).

Tableaux Rules



Soundness and Completeness

- Semantic tableau for propositional logic is sound and complete.
- Soundness:
  - If p1, p2, … $\vdash_{ST}$ q then p1, p2, … $\vDash$ q. i.e. Only proves tautologies.
- Completeness:
  - If p1, p2, … $\vDash$ q then p1, p2, … $\vdash_{ST}$ q. i.e. Can be used to prove all tautologies.

Examples

- $\Phi \vDash \neg p \wedge p$

  $\neg(\neg p \vee p)$

  NOT-AND Rule

  $p \qquad \neg p$

  Not closed □ Consistent

  □ $\Phi \nvDash \neg p \wedge p$

Examples (2)

- $\Phi \vDash \neg(p \wedge q)$ □ $(\neg p \vee \neg q)$
  $\neg$ [$\neg(p \wedge q)$ □ $(\neg p \vee \neg q)$]

NOT-IFF Rule

$\neg(p\wedge q)$ | $(p\wedge q)$
$\neg(\neg p\vee\neg q)$ | $(\neg p\vee\neg q)$

NOT-AND | AND Rule

$\neg p$ | $\neg q$ | p
$\neg(\neg p\vee\neg q)$ | $\neg(\neg p\vee\neg q)$ | q
NOT-OR | NOT-OR | $(\neg p\vee\neg q)$

OR Rule

$\neg p$ | $\neg q$ | p | p
p | p | q | q
$\underline{q}$ | $\underline{q}$ | $\underline{\neg p}$ | $\underline{\neg q}$

Closed □ Inconsistent □ $\Phi \vDash \neg(p \wedge q)$ □ $(\neg p \vee \neg q)$ Tautology

References

- www.student.math.uwaterloo/~se112
- http://www.cogs.susx.ac.uk/users/billk/lectures/lec9.pdf