Transitive Closure of Binary Relation

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• **What is Binary Relation?**

A binary relation \( R \) from the set \( S \) to the set \( T \) is a subset of \( S \times T \), \( R \subseteq S \times T \). If \( S = T \), we say that the relation is a binary relation on \( S \).

• **Properties of Binary Relation**

Let \( R \) be a binary relation on \( S \). Then \( R \) is

- **Reflexive:** iff \( (\forall x), \ (x \in S \rightarrow xR x) \)
- **Symmetric:** iff \( (\forall x)(\forall y), \ (x \in S \wedge y \in S \wedge xR y \rightarrow yR x) \)
- **Anti-symmetric:** iff \( (\forall x)(\forall y), \ (x \in S \wedge y \in S \wedge xR y \wedge yR x \rightarrow x = y) \)
- **Transitive:** iff \( (\forall x) \ (\forall y)(\forall z), \ (x \in S \wedge y \in S \wedge z \in S \wedge xR y \wedge yR z \rightarrow xRz) \)

Some binary relations don’t have these properties.
• **Closures of Binary Relation**

A binary relation $R$ on a set $S$ may not have a particular property such as reflexivity, symmetry, or transitivity. However, it may be possible to extend the relation so that it does have the property.

Extending $R$ means finding a larger subset of $S \times S$ that contains $R$ and which has the desired property. The closure of a relation on $S$ with respect to a property is the smallest such extension that has the desired property.

Commonly used Closures:

- **transitive closure**
- reflexive closure
- symmetric closure
• **Transitive Closure of Binary Relation**

A relation $R^t$ is the transitive closure of a binary relation $R$ if and only if:

1. $R^t$ is transitive,
2. $R \subseteq R^t$, and
3. for any relation $S$, if $R \subseteq S$ and $S$ is transitive, then $R^t \subseteq S$, that is, $R^t$ is the smallest relation that satisfies (1) and (2).

• **How to find Transitive Closure?**

We need to add the minimum number of tuples to $R$, giving us $R^t$, such that if $(a,b)$ is in $R^t$ and $(b,c)$ is in $R^t$, then $(a,c)$ is in $R^t$.  

$$R^t = R \cup \Delta$$

$$(a,b) \in R^t \land (b,c) \in R^t \rightarrow (a,c) \in R^t$$
Graphical Construction of Transitive Closure

Original Relation

Original Relation plus 2-jumps

Original Relation plus 2-jumps, 3-jump
Example of Transitive Closure:

Let $S = \{1, 2, 3\}$.
$R = \{(1,1), (1,2), (1,3), (2,3), (3,1)\}$.

$(2,3) \in R \land (3,1) \in R \rightarrow (2,1) \in R^t$
$(3,1) \in R \land (1,2) \in R \rightarrow (3,2) \in R^t$
$(3,1) \in R \land (1,3) \in R \rightarrow (3,3) \in R^t$
$(2,1) \in R^t \land (1,2) \in R \rightarrow (2,2) \in R^t$ (*Must be done iteratively)

So, $R^t = R \cup \{(2,1), (3,2), (3,3), (2,2)\}$
• Transitive Closure and Composition

Since transitivity is connected to composition, we can also express Transitive Closure using composition.

Theorem:

\[ R^t = R^1 \cup R^2 \cup \ldots \cup R^j \cup \ldots \]

R is a binary relation on A. For \( j \geq 1 \) we define the powers \( R^j \) of \( R \): put \( R^1 = R \) and \( R^{j+1} = R^j \circ R \).

Summary,

• What is Binary Relation?
• The Properties of Binary Relation?
• What is Transitive Closure of Binary Relation?
• How to find Transitive Closure?
• Transitive Closure & Composition