Transitive Closure of Binary Relation

Presented By: Xiaoyang Yu

Revision: 3 October 2002

- ***What is Binary Relation?***

A binary relation R from the set S to the set T is a subset of $S \times T$, $R \subseteq S \times T$. If $S = T$, we say that the relation is a binary relation on S.

- ***Properties of Binary Relation***

Let R be a binary relation on S. Then R is

Reflexive: iff $(\forall x), \ (x \in S \rightarrow xRx)$

Symmetric: iff $(\forall x)(\forall y), \ (x \in S \wedge y \in S \wedge xRy \rightarrow yRx)$

Anti-symmetric: iff $(\forall x)(\forall y), (x \in S \wedge y \in S \wedge xRy \wedge yRx$

$\rightarrow x = y)$

Transitive: iff $(\forall x)(\forall y)(\forall z), (x \in S \wedge y \in S \wedge z \in S$

$\wedge xRy \wedge yRz \rightarrow xRz)$

Some binary relations don’t have these properties.

• *Closures of Binary Relation*

A binary relation R on a set *S* may not have a particular property such as reflexivity, symmetry, or transitivity. However, it may be possible to extend the relation so that it does have the property.

Extending R means finding a <u>larger subset</u> of $S \times S$ that contains R and which has the desired property. The closure of a relation on *S* with respect to a property is the <u>smallest such extension</u> that has the <u>desired property</u>.

Commonly used Closures:

transitive closure

reflexive closure

symmetric closure

- ***Transitive Closure of Binary Relation***

A relation R^t is the transitive closure of a binary relation R if and only if:

(1) R^t is transitive,
(2) $R \subseteq R^t$, and
(3) for any relation S, if $R \subseteq S$ and S is transitive, then $R^t \subseteq S$, that is, R^t is the smallest relation that satisfies (1) and (2).

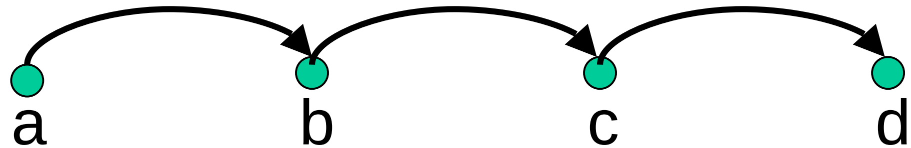
- ***How to find Transitive Closure?***

We need to add the minimum number of tuples to R, giving us R^t, such that if (a,b) is in R^t and (b,c) is in R^t, then (a,c) is in R^t.

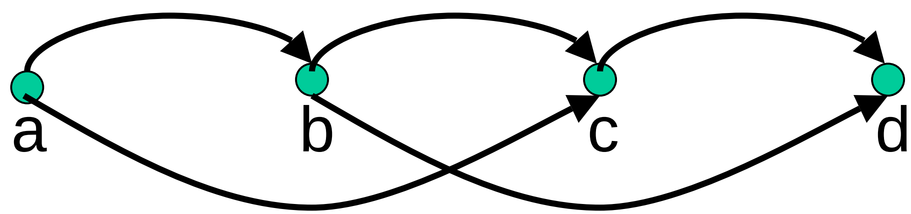
$$R^t = R \cup \Delta$$

$$(a,b) \in R^t \wedge (b,c) \in R^t \rightarrow (a,c) \in R^t$$

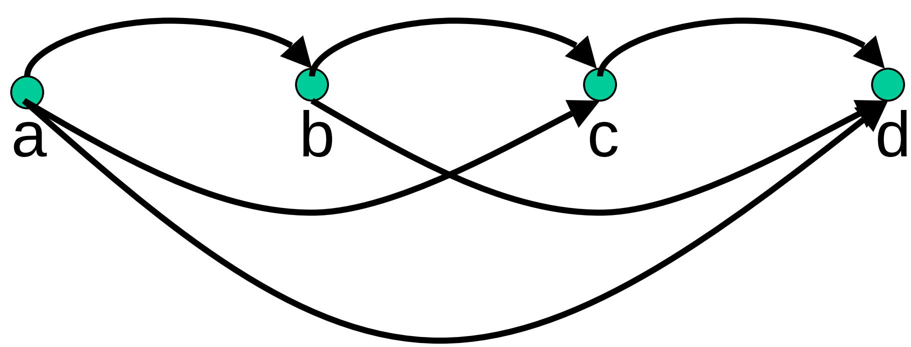
Graphical Construction of Transitive Closure



Original Relation



Original Relation plus 2-jumps



Original Relation plus 2-jumps, 3-jump

Example of Transitive Closure:

Let $S = \{1, 2, 3\}$.

$R = \{(1,1), (1,2), (1,3), (2,3), (3,1)\}$.

$(2,3) \in R \wedge (3,1) \in R \rightarrow (2,1) \in R^t$

$(3,1) \in R \wedge (1,2) \in R \rightarrow (3,2) \in R^t$

$(3,1) \in R \wedge (1,3) \in R \rightarrow (3,3) \in R^t$

$(2,1) \in R^t \wedge (1,2) \in R \rightarrow (2,2) \in R^t$ (*Must be done iteratively)

So, $R^t = R \cup \{(2,1), (3,2), (3,3), (2,2)\}$

- **Transitive Closure and Composition**

Since transitivity is connected to composition, we can also express Transitive Closure using composition.

Theorem:

$$R^t = R^1 \cup R^2 \cup \ldots \cup R^j \cup \ldots$$

R is a binary relation on A. For $j \geq 1$ we define the powers R^j of R: put $R^1 = R$ and $R^{j+1} = R^j \circ R$.

Summary,

- What is Binary Relation?
- The Properties of Binary Relation?
- What is Transitive Closure of Binary Relation?
- How to find Transitive Closure?
- Transitive Closure & Composition