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02 Mathematical Models

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Kinds of Mathematical Models

- There are many kinds of mathematical models

- The major categories are:

  1. Mathematical models constructed from basic mathematical objects (such as sets, functions, and relations)
  2. Mathematical structures (collections of objects related in certain ways)
  3. Axiomatic theories (which axiomatically specify a set of mathematical structures)
  4. Abstract machines for performing computations

Sets

- A **set** is a collection of objects with a **membership** relation (usually represented by $\in$)

- Almost any mathematical concept can be expressed in terms of set and membership

- There are various paradoxes that involve sets that are "too big"
  - Example: **Russell's Paradox** involving the set of all sets that do not contain themselves

- **Zermelo-Fraenkel (ZF) set theory** is a first-order formalization of set theory that is intended to avoid the set-theoretic paradoxes
  - It is widely used as a lingua franca for mathematics

Set Concepts

- Basic properties: membership, subset, cardinality
- Basic operations:
  - union, intersection, complement, difference, symmetric-difference
  - cartesian product (product), disjoint union (sum)
  - power set
- Special sets: the emptyset, universal sets, tuples, functions, relations, ordinals, cardinals

Functions

- There are two definitions of a function:
  1. A **function** is a rule $f : I \to O$ that associates members of I (inputs) with members of O (outputs)
     - Every input is associated with at most one output
     - Some inputs may not be associated with an output

       Example: f : **Z** $\to$ **Q** where $x \mapsto 1/x$
  2. A **function** is a set $F \subset I \times O$ such that if $(x, y), (x, y') \in F$, then $y = y'$

- Each function f has a **domain** $D \subseteq I$ and a **range** $R \subseteq O$

- A set can be represented as a special kind of function (e.g., as a **unary predicate**, a **characteristic function**, or an **indicator**)

Function Concepts

- Basic properties:
  - partial, total
  - arity (0, 1, $n \geq 2$, multiary)
  - injective, surjective, bijective
  - image, reverse image

- Basic operations: composition, restriction, inverse

- Special functions: the empty function, identity functions, choice functions

Relations

- An n-ary **relation** is a set $R \subseteq A_1 \times \cdots \times A_n$ ($n \geq 1$)
  - Any set can be considered as a unary relation
  - Any nonunary relation can be considered as a binary relation
- Functions are considered as special relations
  - An n-ary function $f : A_1 \times \cdots \times A_n \to B$ is identified with the corresponding $(n+1)$-ary relation $R_f \subseteq A_1 \times \cdots \times A_n \times B$
- An n-ary relation can be represented by an n-ary predicate

Relation Concepts

- Basic relation properties:
  - reflexive, symmetric, transitive

- Basic relation operations:
  - domain, range
  - composition, inverse

- Special relations: the empty relation, universal relations, equivalence relations

Example: Program Specifications

Various kinds of functions and relations can used to specify computer programs:

- **Input/output specification**
  - A function $f : I \to O$ that maps inputs to outputs
  - A relation $R \subseteq I \times O$ that relates inputs and outputs
- **Before/after specification**
  - A function $f : I \times S \to O \times S$ that maps inputs and before-states to outputs and after-states
  - A relation $R \subseteq I \times S \times O \times S$ that relates inputs, before-states, outputs, and after-states
- **Trace specification**
  - A function $f : I \times S^* \to O \times S^*$ that maps inputs and before-traces to outputs and after-traces
  - A relation $R \subseteq I \times S^* \times O \times S^*$ relates inputs, before-traces, outputs, and after-traces

Mathematical Structures

- A **mathematical structure** consists of:
  1. A set of elements called the **domain**
  2. A set of distinguished elements, functions, and relations called the **signature** that impose a structure on the set of elements

- The signature usually determines a **language** for describing and making assertions about the elements of the domain

Examples of Mathematical Structures

- Orders (e.g., pre, partial, linear, well)
- Graphs
- Lattices and boolean algebras
- Algebraic structures (e.g, monoids, groups, rings, fields, modules, vector spaces)
- Number systems (e.g, integer, rational, real, complex, hyperreal, ordinal, cardinal, surreal)
- Data structures used in Computer Science (e.g, arrays, stacks, queues)

Axiomatic Theories

- **Axiomatic theory** = formal language + set of axioms
- **Language**: vocabulary for describing and making assertions about objects and their properties
- **Axioms**: assumptions about the objects and properties
  - Specify a set of mathematical structures (called the **models** of the theory)
  - Basis for proving theorems
- Benefits:
  - **Conceptual clarity**: inessential details are omitted
  - **Generality**: theorems hold in all models

Abstract Machines

- An **abstract machine** or **automaton** is an abstract device for performing computations

- A **universal machine** is a machine that can compute any computable function; examples:
  - Turing machine
  - Unlimited register machine

- A **state machine** is a machine that computes in a stepwise fashion in which each step includes receiving input, producing output, and changing state; examples:
  - Finite automaton
  - Pushdown automaton
  - Finite state machine

Turing Machines (Alan Turing, 1936)

A **(deterministic) Turing machine** M consists of the following components:

1. A two-way infinite **tape** composed of cells

2. A **tape head** that can read a cell and then (a) write a symbol in it, (b) move left, or (c) move right

3. A **finite state control mechanism**

4. Fixed finite set $\{s_1, \ldots, s_m\}$ of **tape symbols**

5. Fixed finite set $\{q_1, \ldots, q_n\}$ of **states**

6. A **transition relation** consisting of a finite set T of quadruples of the form (q_i, s_j, s_k, q_l), (q_i, s_j, L, q_l), or (q_i, s_j, R, q_l) such that (a) $1 \leq i, l \leq n$, (b) $0 \leq j, k \leq m$, and (c) for all (q_i, s_j), there is at most one $(q_i, s_j, \alpha, \beta) \in T$

Finite State Machines

A **finite state machine** M consists of the following components:

1. A fixed finite set S of **states** including an **initial state**

2. A fixed (possibly infinite) set I of **inputs**

3. A fixed (possibly infinite) set O of **outputs**

4. A **next state** relation ns $\subseteq S \times I \times S$

5. An **output** relation out $\subseteq S \times I \times O$

M is **deterministic** if the relations are functions, i.e., ns : $S \times I \to S$ and out : $S \times I \to O$