CAS 701 Fall 2002

03 Review of Logic

Instructor: W. M. Farmer

Revised: 24 September 2002

What is a Logic?

- Informally, a logic is a system of reasoning

- Formally, a **logic** is a family of **formal languages** with:

  1. A common syntax
  2. A common semantics
  3. A notion of **logical consequence**

- A logic may include a **formal system** for **proving** that a given formula is a logical consequence of a given set of formulas

- Examples:
  - Propositional logic
  - First-order logic
  - Simple type theory (higher-order logic)

Language Syntax

- A language defines a collection of **expressions** formed from:
  - **Variables**
  - **Constants** (nonlogical constants)
  - **Constructors** (logical constants)

- Two kinds of expressions:
  - **Terms**: Denote objects or values
  - **Formulas**: Make assertions about objects or values

- Some languages have constructors that bind variables (e.g., $\forall$, $\exists$, λ, I, ϵ, $\{\ \mid\ \}$)

Language Semantics

- A **model** M for a language L is a pair (D, V) where:
  1. D is a set of values called the **domain** that includes the truth values `true` and `false`
  2. V is a function from the expressions of L to D called the **valuation function**

- M **satisfies** a formula φ of L, written $M \models \varphi$, if $V(\varphi) = \texttt{true}$

- M **satisfies** a set Σ of formulas of L, written $M \models \Sigma$, if M satisfies each $\varphi \in \Sigma$

- Σ is **satisfiable** if there exists some model for L that satisfies Σ

- φ is **valid**, written $\models \varphi$, if every model for L satisfies φ

- φ is a **logical consequence** of Σ, written $\Sigma \models \varphi$, if every model for L that satisfies Σ also satisfies φ

Hilbert-Style Formal System

- A **Hilbert-style formal system H** for a language L consists of:
  1. A set of formulas of L called **logical axioms**
  2. A set of **rules of inference**

- A **proof** of φ from Σ in H is a finite sequence $\psi_1, \ldots, \psi_n$ of formulas of L with $\psi_n = \varphi$ such that each ψ_i is either a logical axiom, a member of Σ, or follows from earlier ψ_j by one of the rules of inference

- φ is **provable** from Σ in H, written $\Sigma \vdash_H \varphi$, if there is a proof of φ from Σ in H

- φ is a **theorem** in H, written $\vdash_H \varphi$, if φ is provable from $\emptyset$ in H

- Σ is **consistent** in H if not every formula is provable from Σ in H

Kinds of Formal Systems

- Hilbert style
- Symmetric sequent (Gentzen)
- Asymmetric sequent
- Natural deduction (Quine, Fitch, Berry)
- Semantic tableau (Beth, Hintikka)
- Resolution (J. Robinson)

Soundness and Completeness

- Let F be a formal system for a language L

- F is **sound** if

$$\Sigma \vdash_F \varphi \quad \text{implies} \quad \Sigma \models \varphi$$

- F is **complete** if

$$\Sigma \models \varphi \quad \text{implies} \quad \Sigma \vdash_F \varphi$$

- **Gödel's Completeness Theorem**: There is a sound and complete formal system F for first-order logic
  - **Corollary**: Σ is satisfiable iff Σ is consistent in F

Theories

- A **theory** is a pair $T = (L, \Gamma)$ where:
  1. L is a language (the **language** of T)
  2. Γ is a set of formulas of L (the **axioms** of T)

- M is a **model** of T, written $M \models T$, if $M \models \Gamma$

- φ is a **valid** in T, written $T \models \varphi$, if $\Gamma \models \varphi$

- φ is a **theorem** of T in F, written $T \vdash_F \varphi$, if $\Gamma \vdash_F \varphi$

- T is **satisfiable** if Γ is satisfiable

- T is **consistent** in F if Γ is consistent in F

Complete Theories

- Three possibilities:
  1. φ is valid in T
  2. $\neg\varphi$ is valid in T
  3. Neither φ nor $\neg\varphi$ is valid in T
     - Hence, some model of T satisfies φ
     - Hence, some model of T satisfies $\neg\varphi$

- A theory $T = (L, \Gamma)$ is **complete** if, for all formulas φ of L, $T \models \varphi$ or $T \models \neg\varphi$
  - Notice that an unsatisfiable theory is always complete

- **Gödel's Incompleteness Theorem**: Let $T = (L, \Gamma)$ be a satisfiable theory such that Γ is a recursive set. If T is sufficiently "rich", then T is incomplete.

Semantics vs. Syntax

Semantics	**Syntax**
φ is valid $\models \varphi$	φ is a theorem in F $\vdash_F \varphi$
φ is valid in T $T \models \varphi$	φ is a theorem of T in F $T \vdash_F \varphi$
T is satisfiable	T is consistent in F

- By Gödel's Completeness Theorem, the semantic and syntactic notions for first-order logic are equivalent

- The problem whether or not $T \models \varphi$ is true can be solved by either:

  1. **Proof**: Showing $T \vdash_F \varphi$ for some sound formal system F or
  2. **Counterexample**: Showing $M \models \neg\varphi$ for some model M of T

Theory Extensions

- Let $T = (L, \Gamma)$ and $T' = (L', \Gamma')$ be theories

- T' is an **extension** of T, written $T \leq T'$, if:
  1. $L \leq L'$ (L' is an extension of L)
  2. $\Gamma \subseteq \Gamma'$

- T' is a **conservative extension** of T, written $T \trianglelefteq T'$, if:
  1. $T \leq T'$
  2. For all formulas φ of L, if $T' \models \varphi$, then $T \models \varphi$.

Theories as Logics

- A logic is identified by its set of theories
- A theory T can be viewed as the logic that is identified by the set of extensions of T
- Examples of theories that are often used as logics:
  - Peano arithmetic
  - Real arithmetic (theory of complete ordered fields)
  - Theory of real closed fields
  - Zermelo-Fraenkel (ZF) set theory
  - Von-Neumann-Bernays-Gödel (NBG) set theory