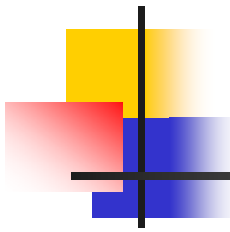


Ordinal Numbers

Ramez Mousa

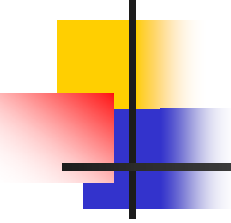
Oct. 14, 2004

CAS 701



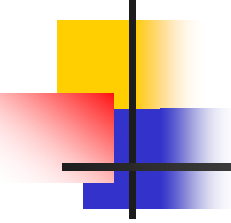
Agenda

- Introduction
- Basic Definitions
- Ordinal Numbers
- Arithmetic of Ordinal Numbers
- Final Remarks
- References



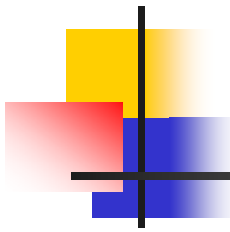
Introduction

- A natural number can be used for 2 purposes:
 - Describe the size of a set
 - Describe the position of an element in a sequence
- In the finite world, these 2 concepts coincide_[vi]
- In the infinite world, the two concepts need to be distinguished
 - Size aspect leads to Cardinal Numbers
 - Position aspect leads to Ordinal Numbers
- Thus, with respect to the finite world, ordinal numbers and cardinal numbers are the same_[vi]



Basic Definitions

- Well Ordered Set: A totally ordered set $(A, \leq)$ is well ordered if and only if every nonempty subset of A has a least element
 - Set of nonnegative integers is well ordered
 - Set of integers is not well ordered_[v]
- Order Isomorphic: 2 totally ordered sets $(A, \leq)$ and $(B, \leq)$ are order isomorphic if and only if there is a bijection from A to B such that
 - For all $a_1, a_2 \in A$, $a_1 \leq a_2$ if and only if $f(a_1) \leq f(a_2)$ _[v]
- Proper Class: A Class is an arbitrary collection of elements. Classes which are not sets are called proper classes_[iv]



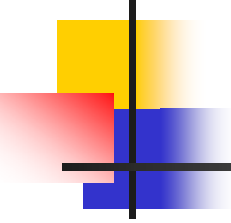
Ordinal Numbers

- Informally, used to denote the position of an element in an ordered sequence_[vi]
- Formally, it is one of the numbers in Georg Cantor's extension of the whole numbers_[vi]
- Defined as the order type of a well ordered set_[v]
- Order type of a well ordered set M , is obtained by counting elements of M in correct order_[ii]
- Therefore, given a finite set, can determine its ordinal number by counting order type
- A well ordered finite set with k elements has k as order type and ordinal number



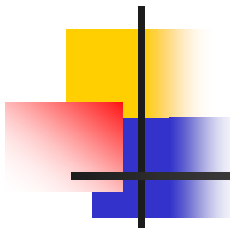
Ordinal Numbers (cont'd)

Set	Ordinal
$\{ \}$	0
$\{0\}$	1
$\{0, 1\}$	2
$\{0, 1, 2\}$	3
...	...
$\{0, 1, 2, \dots\}$	ω
$\{0, 1, 2, \dots, 0\}$	$\omega+1$
...	...
$\{0, 1, 2, \dots, 0, 1, 2, \dots\}$	$\omega+\omega$
$\{0, 1, 2, \dots, 0, 1, 2, \dots, 0\}$	$\omega+\omega+1$



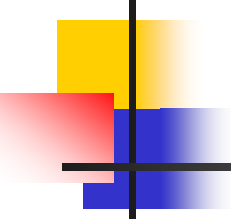
Ordinal Numbers (cont'd)

- Problem occurs when given set is infinite
 - Example: set of nonnegative integers $\{0, 1, 2 \dots\}$
 - Can not determine order type of set by counting
- ω is order type of set of nonnegative integers_[v]
- ω is smallest ordinal number greater than the ordinal number of whole numbers_[v]
- Next ordinal after ω is $\omega+1$
- Ordinal numbers them self form a well ordered set_[vi]
- Ordinal numbers are: $0, 1, 2, \dots, \omega, \omega+1, \omega+2, \dots, \omega+\omega, \omega+\omega+1, \dots$ _[v]



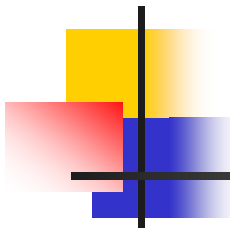
Ordinal Numbers (cont'd)

- Given a well ordered set $(A, \leq)$ with ordinal k , the set of all ordinals $< k$ is order-isomorphic to A
- Define an ordinal as the set of all ordinals less than itself.
- Example, 0 as $\{ \}$, 1 as $\{0\}$, 2 as $\{0, 1\}$, 3 as $\{0, 1, 2\}$, k as $\{0, 1, \dots, k-1\}_{[v]}$
- Every well ordered set is order-isomorphic to one and only one ordinal_[v]
- Can also determine next larger ordinal $k+1$ with the union operation $k \cup \{k\}_{[v]}$
- No largest ordinal_[vi]
- Collection of all ordinals form a proper class_[v]



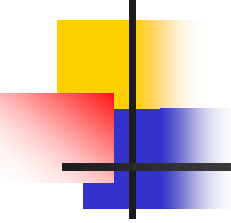
Ordinal Numbers (cont'd)

- Mathematician John von Neumann defined a set A to be an ordinal number if and only if:
 - If a and b are members of A , then either $a=b$, a is a member of b , b is a member of a (strictly well ordered with respect to subset relation)_[v]
 - If a_1 is a member of A , then a_1 is a proper subset of A _[v]
- Example, given ordinal 2 represented as $\{0,1\}$
 - $\{0, 1\}$ has 2 members: 0 and 1 represented as $\{ \}$ and $\{0\}$ respectively
 - Well ordered since $\{ \}$ is a member of $\{0\}$
 - Since $\{ \}$ and $\{0\}$ are members of ordinal 2 we have $\{ \}$ and $\{0\}$ are proper subsets of 2



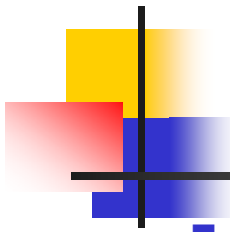
Arithmetic of Ordinal Numbers

- Adding ordinals $S+T$ forms a new well ordered set that is order-isomorphic to ordinal $S+T$ _[vi]
- Addition of finite ordinals similar to integers_[vi]
- Addition of transfinite ordinals is a bit tricky
 - Add $3+\omega$, get $\{0, 1, 2, 0', 1', 2' \dots\}$
 - Relabeling the above set, we get ω itself
 - Now add $\omega+3$, get $\{0, 1, 2, \dots, 0', 1', 2'\}$
 - $\omega+3 > 3+\omega$, because as you pair the numbers in the first set with the numbers in the second set, you never reach the extra numbers 0, 1, 2 at the end_[iii]
 - $\omega+3$ has a largest element, while $3+\omega$ does not
- Thus addition is associative but not commutative_[vi]



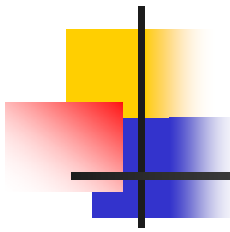
Arithmetic of Ordinal Numbers (cont'd)

- To multiply 2 ordinals S and T :
 - Write down the well ordered set T and replace each of its elements with a different copy of S
- Multiplying ordinals S and T forms a new well ordered set that is order-isomorphic to $S * T$ _[vi]
- Multiplying transfinite ordinals is also a bit tricky:
 - Multiply $\omega * 2$, get $\{0, 1, 2, \dots, 0', 1', 2', \dots\}$
 - Observe that $\omega * 2 = \omega + \omega$
 - Now multiply, $2 * \omega$, get $\{0, 1, 0', 1', 0'', 1'', \dots\}$
 - Relabeling, we get ω . Thus $2 * \omega = \omega$
- Like addition, multiplication of ordinals is associative but not commutative _[vi]



Final Remarks

- For finite sets, can determine ordinal by counting order type
- For infinite sets, impossible to determine order type, thus denote omega
- Ordinal Numbers form a well ordered set
- Can write ordinal numbers as set of all ordinals less than itself
- Addition and multiplication of ordinals form new ordinals $S+T$ and $S*T$
- Addition and multiplication of ordinals are associative but not commutative_[vi]
- *Ordinal numbers are not so confusing after all!!*



References

- [i] Conway, J. H. and Guy, R. K. "Cantor's Ordinal Numbers." In The Book of Numbers. New York: Springer-Verlag, 1996.
- [ii] Rucker, Rudy. "Transfinite Ordinals". October 12, 2004. <<http://www.anselm.edu/homepage/dbanach/infin.htm>>
- [iii] Savard, John J. G. "Infinite Ordinal". October 12, 2004. <<http://home.ecn.ab.ca/~jsavard/math/inf01.htm>>
- [iv] Schunemann, Ulf. "Sets And Numbers". October 12, 2004. <<http://web.cs.mun.ca/~ulf/gloss/sets.html>>
- [v] Weisstein, Eric W. "Ordinal Number". October 12, 2004. <<http://mathworld.wolfram.com/OrdinalNumber.html>>
- [vi] Wikipedia The Free Encyclopedia. "Ordinal Number". October 12, 2004. <http://en.wikipedia.org/wiki/Ordinal_number>

Thank You For Your Time!!!