Ordinal Numbers

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Agenda

- Introduction
- Basic Definitions
- Ordinal Numbers
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Introduction

- A natural number can be used for 2 purposes:
  - Describe the size of a set
  - Describe the position of an element in a sequence
- In the finite world, these 2 concepts coincide[vi]
- In the infinite world, the two concepts need to be distinguished
  - Size aspect leads to Cardinal Numbers
  - Position aspect leads to Ordinal Numbers
- Thus, with respect to the finite world, ordinal numbers and cardinal numbers are the same[vi]

Basic Definitions

- Well Ordered Set: A totally ordered set (A, ≤) is well ordered if and only if every nonempty subset of A has a least element
  - Set of nonnegative integers is well ordered
  - Set of integers is not well ordered[v]
- Order Isomorphic: 2 totally ordered sets (A, ≤) and (B, ≤) are order isomorphic if and only if there is a bijection from A to B such that
  - For all $a_1, a_2 \in A$, $a_1 \leq a_2$ if and only if $f(a_1) \leq f(a_2)$[v]
- Proper Class: A Class is an arbitrary collection of elements. Classes which are not sets are called proper classes[iv]

Ordinal Numbers

- Informally, used to denote the position of an element in an ordered sequence[vi]
- Formally, it is one of the numbers in Georg Cantor's extension of the whole numbers[vi]
- Defined as the order type of a well ordered set[v]
- Order type of a well ordered set M, is obtained by counting elements of M in correct order[ii]
- Therefore, given a finite set, can determine its ordinal number by counting order type
- A well ordered finite set with k elements has k as order type and ordinal number

Ordinal Numbers (cont'd)

Set	Ordinal
{ }	0
{0}	1
{0, 1}	2
{0, 1, 2}	3
…	…
{0, 1, 2 …}	ω
{0, 1, 2 …, 0}	$\omega+1$
…	…
{0, 1, 2, …, 0, 1, 2, ..}	$\omega+\omega$
{0, 1, 2, …, 0, 1, 2, .., 0}	$\omega+\omega+1$

Ordinal Numbers (cont'd)

- Problem occurs when given set is infinite
  - Example: set of nonnegative integers {0, 1, 2 …}
  - Can not determine order type of set by counting
- ω is order type of set of nonnegative integers [v]
- ω is smallest ordinal number greater than the ordinal number of whole numbers [v]
- Next ordinal after ω is $\omega+1$
- Ordinal numbers them self form a well ordered set [vi]
- Ordinal numbers are: $0, 1, 2, \ldots, \omega, \omega+1, \omega+2, \ldots, \omega+\omega, \omega+\omega+1, \ldots$ [v]

Ordinal Numbers (cont'd)

- Given a well ordered set (A, ≤) with ordinal k, the set of all ordinals < k is order-isomorphic to A
- Define an ordinal as the set of all ordinals less that itself.
- Example, 0 as { }, 1 as {0}, 2 as {0, 1}, 3 as {0, 1, 2}, k as {0, 1, …, k-1}[v]
- Every well ordered set is order-isomorphic to one and only one ordinal[v]
- Can also determine next larger ordinal k+1 with the union operation k U {k}[v]
- No largest ordinal[vi]
- Collection of all ordinals form a proper class[v]

Ordinal Numbers (cont'd)

- Mathematician John von Neumann defined a set A to be an ordinal number if and only if:
  - If a and b are members of A, then either a=b, a is a member of b, b is a member of a (strictly well ordered with respect to subset relation)[v]
  - If a_1 is a member of A, then a_1 is a proper subset of A[v]
- Example, given ordinal 2 represented as {0,1}
  - {0, 1} has 2 members: 0 and 1 represented as { } and {0} respectively
  - Well ordered since { } is a member of {0}
  - Since { } and {0} are members of ordinal 2 we have { } and {0} are proper subsets of 2

Arithmetic of Ordinal Numbers

- Adding ordinals S+T forms a new well ordered set that is order-isomorphic to ordinal S+T[vi]
- Addition of finite ordinals similar to integers[vi]
- Addition of transfinite ordinals is a bit tricky
  - Add $3+\omega$, get $\{0, 1, 2, 0', 1', 2' \ldots\}$
    - Relabeling the above set, we get ω itself
  - Now add $\omega+3$, get $\{0, 1, 2, \ldots, 0', 1', 2'\}$
    - $\omega+3 > 3+\omega$, because as you pair the numbers in the first set with the numbers in the second set, you never reach the extra numbers 0, 1, 2 at the end[iii]
  - $\omega+3$ has a largest element, while $3+\omega$ does not
- Thus addition is associative but not commutative[vi]

Arithmetic of Ordinal Numbers (cont'd)

- To multiply 2 ordinals S and T:
  - Write down the well ordered set T and replace each of its elements with a different copy of S
- Multiplying ordinals S and T forms a new well ordered set that is order-isomorphic to S*T[vi]
- Multiplying transfinite ordinals is also a bit tricky:
  - Multiply $\omega*2$, get $\{0, 1, 2, \ldots, 0', 1', 2', \ldots\}$
    - Observe that $\omega*2 = \omega+\omega$
  - Now multiply, $2*\omega$, get $\{0, 1, 0', 1', 0'', 1'', \ldots\}$
    - Relabeling, we get ω. Thus $2*\omega = \omega$
- Like addition, multiplication of ordinals is associative but not commutative[vi]

Final Remarks

- For finite sets, can determine ordinal by counting order type
- For infinite sets, impossible to determine order type, thus denote omega
- Ordinal Numbers form a well ordered set
- Can write ordinal numbers as set of all ordinals less than itself
- Addition and multiplication of ordinals form new ordinals S+T and S*T
- Addition and multiplication of ordinals are associative but not commutative[vi]
- *Ordinal numbers are not so confusing after all!!*

References

- [i] Conway, J. H. and Guy, R. K. "Cantor's Ordinal Numbers." In The Book of Numbers. New York: Springer-Verlag, 1996.
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- [iv] Schunemann, Ulf. “Sets And Numbers”. October 12, 2004. <http://web.cs.mun.ca/~ulf/gloss/sets.html>
- [v] Weisstein, Eric W. "Ordinal Number“. October 12, 2004. <http://mathworld.wolfram.com/OrdinalNumber.html>
- [vi] Wikipedia The Free Encyclopedia. “Ordinal Number”. October 12, 2004. <http://en.wikipedia.org/wiki/Ordinal_number>

Thank You For Your Time!!!