Schroeder-Bernstein Theorem

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Statement of Theorem

Theorem: Given two sets A, B and two injective functions

$$f : A \longrightarrow B$$

$$g : B \longrightarrow A$$

there exists a bijective function

$$h : A \longrightarrow B$$

Also we can claim it like this:

if $|S| \leq |T|$ and $|T| \leq |S|$, then $|S|=|T|$

$|S|$ is the cardinality of the set S.

Cardinal Comparison

For any sets *A* and *B*, their cardinal numbers satisfy $|A| \leq |B|$ iff there is a one-to-one function *f* from *A* into *B*

(Rubin 1967, p. 266; Suppes 1972, pp. 94 and 116).

It is easy to show this satisfies the reflexive and transitive axioms of a partial order. However, it is difficult to show the antisymmetry property, whose proof is known as the Schröder-Bernstein theorem.

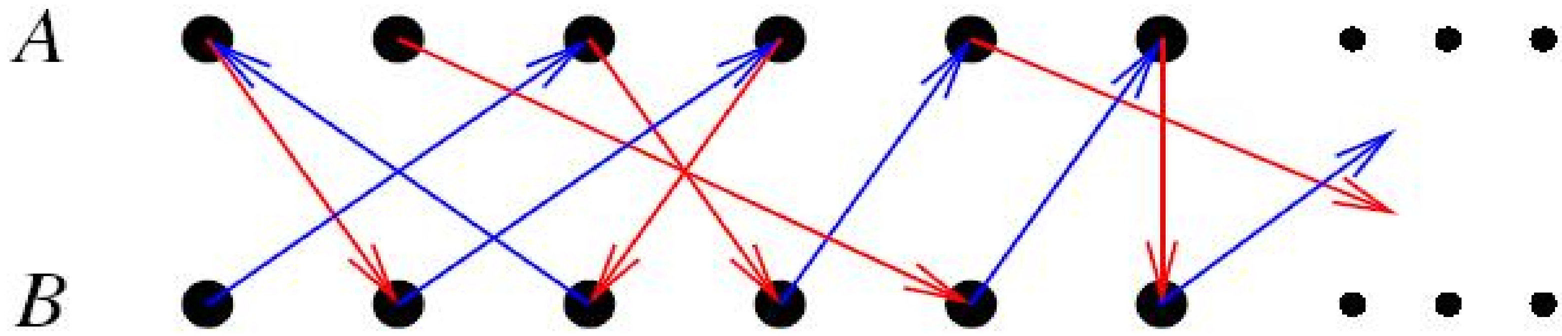
Trichotomy axiom

It states that any two elements, x and y, must have exactly one of the following relationships:

$x < y$, $x = y$ or $x > y$

The "trichotomy" theorem is the definition of a total order. It can and is and must be proven in both the case of set theory, and the real numbers. The Cantor-Bernstein theorem is *exactly* the proof in the case of sets.

Informal Proof

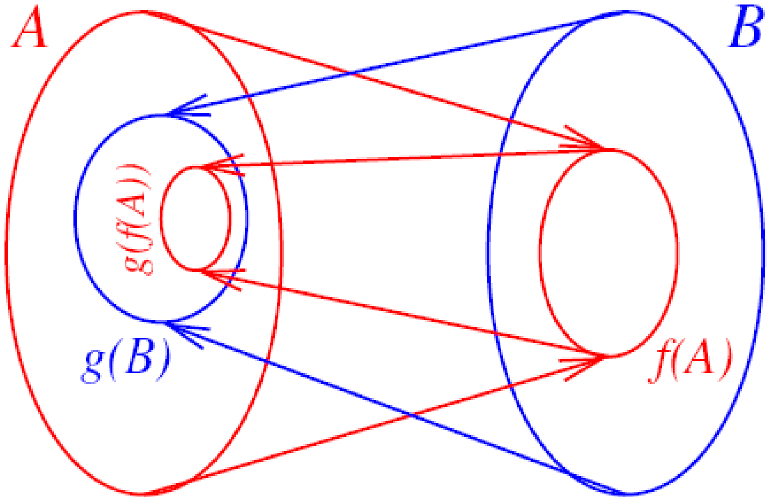


- f represented by $\swarrow \downarrow \searrow$, g represented by $\nwarrow \uparrow \nearrow$.
- Every point has 1 arrow out and at most 1 arrow in.
- For any $a \in A$, say that a is B-stopping if making *backward* jumps starting from a ends up in B.
- Define $h(a) = \begin{cases} g^{-1}(a) & \text{if } a \text{ is } B\text{-stopping.} \\ f(a) & \text{otherwise} \end{cases}$

Reducing the Theorem into Lemma

Lemma: Given two sets A,B with B⊆A, if there exists an injective function f : A → B there exists a bijective function

h : A → B.



Why the lemma implies the theorem

1. Since f and g are injections, $g \circ f$ is an injection from A to g(B). Also $g(B) \subseteq A$, and the conditions of the lemma are now satisfied, so there must exist a bijection h from A to g(B).

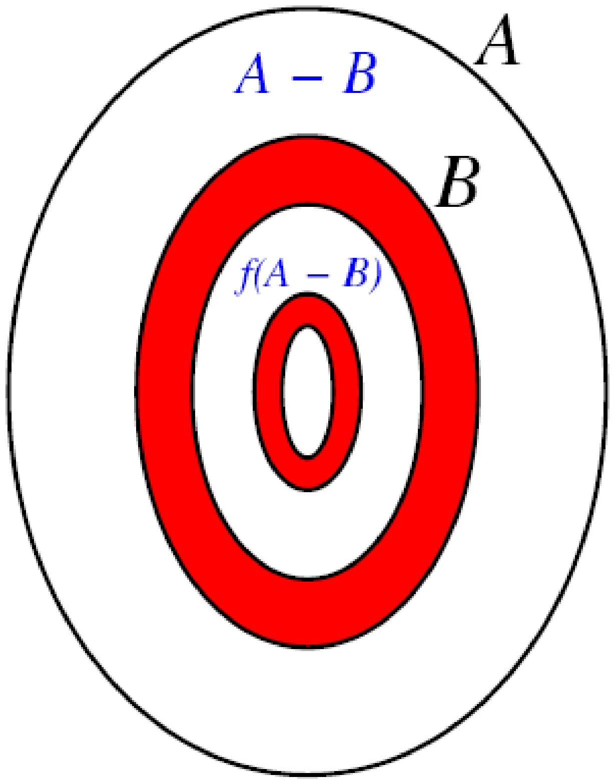
2. It is given that g is an injection from B to A, so g is a bijection from B to g(B). A bijection has an inverse that is a bijection, and so g^{-1} is a bijection from g(B) to B.

3. The composition of two bijections is a bijection, and so $g^{-1} \circ h$ is a bijection from A to B.

Proving the Lemma (1)

Define $X = \cup_{n\geq 0} f^n(A\text{-}B)$

$$H(x) = \begin{cases} f(x) & x \in X \\ x & x \notin X \end{cases}$$



Proving the Lemma (2)

Proof of Claim:

h **is injective:** Suppose $h(x) = h(y)$. If $x , y \in X$ then $x = y$ since f is injective. If $x, y \notin X$ then $x = y$ by the definition of h.

Finally, $f(X) \subset X$ so we cannot have precisely one of x, y in X.

h **is surjective:** Suppose we have $y \in B$. If $y \in X$, then $y \in f^n(A-B)$ for some $1 \leq n$. Therefore, there exists an $x \in f^{n-1}(A-B) \subseteq X$ satisfying $h(x) = f(x) = y$. If $y \notin X$, then $H(y) = y$ by the definition of h.

What is this theorem for

This theorem provides another method for proving cardinal equality. Mapping two sets into each other is often easier than finding a perfect 1-1 correspondence.

For instance, the integers map into the rationals in the obvious way, and rationals map into integers by sending the fraction a/b (lowest terms) to 2^a3^b. (Send -a/b to -2^a3^b, and send 0 to 0.) That's it. The integers and the rationals are the same size.

Some useful links

http://www.mathreference.com/set-card,sbt.html

http://books.pdox.net/Lecture%20Notes/Real%20Analysis%20Notes.pdf

http://www.tfproject.org/tfp/archive/index.php/t-27261.html

http://math.dartmouth.edu/~doyle/docs/three/three.pdf

http://metamath.planetmirror.com/mpegif/sbth.html