Monadic FOL is Decidable

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- What is Monadic FOL?

  L=(Φ, Φ, P) P is a monadic (one-place, unary) predicate which only takes one parameter. Eg. P(x)

- Is it Decidable?

  First-Order logic is undecidable (Church 1936)

  Monadic FOL is a special case. Answer is Yes.

- An intuitive example

  $\exists x.F(x) \wedge \exists x.G(x)$

  $\therefore \exists x.(F(x) \wedge G(x))$

> Theorem 1: (Löwenheim-Skolem 1915)
>
> If S is a monadic sentence which is satisfiable, then S is true in some interpretation whose domain contains at most $2^{k}\cdot r$ members, k being the number of monadic predicate letters and r being the number of variables

Proof Sketch

- Part 1: Proof of decidability of monadic FOL from theorem 1
- Part 2: Proof of theorem 1

Part 1: Proof of decidability of monadic FOL from theorem 1

1. Associate S with a quantifier-free sentence S*, which is satisfiable iff S is.
   i) Find subformula H for S. E.g $S=\forall x.F(x) \vee \exists y.G(y)$ H:F(x), G(y), $\forall x.F(x)$…
   ii) Inductively associate a quantifier-free H* with each H:
      a) If H is atomic, H*=H;
      b) If H is a truth-functional compound, H* is the same compound
      c) If $H=\exists vF$, $H=F(a_1) \vee F(a_2) \vee \ldots \vee F(a_m)$ $m= 2^k*r$
      d) If $H=\forall vF$, $H=F(a_1) \wedge F(a_2) \wedge \ldots \wedge F(a_m)$
   iii) S and S* has the same truth value in the same interpretation
2. S* is easy to decide the validity. So S is decidable.

Proof Sketch

Part 2: Proof of theorem 1

Suppose M is model of S whose domain is D

1. For each d in D, let s(d)=<j_1,…,j_k>, where, for each I between 1 and k, j_i=1 or 0 according as M specifies that P_i is true or false of d. 2^k such sequence s(d)
2. C is similar to d iff s(c)=s(d). Similarity is an equivalence relation. Each d in D belongs to a unique equivalence class. At most 2^k equivalence classes
3. Construct a model M* of S whose domain has at most 2^k*r members
   i) Form a set E: Choose from each equivalence class r members; if there are fewer than r members in the class, take all of the members
   ii) E contains at most 2^k*r members
   iii) M* is the interpretation whose domain is E, and which specifies that P_i to be true of c iff M specifies that P_i is to be true of any c in E
4. M(S)=True, we need to see that M(S)=M*(S)

Proof Sketch

Part 2: Proof of theorem 1 (‘cond)

We need to prove M(S)=M*(S)

Lemma 1:

Suppose G is a subsentence of S, that d_1,…d_n is a sequence of elements of D, that e_1…e_n is a sequence of elements of E, and that d_1,…d_n is exactly like e_1,…e_n. Then G is true in $M_{d1…dn}$ iff G is true in $M^*_{e1…en}$

Subsentence: either a subformula of S or that can be obtained from subformula by substituting names

Exactly likeness: c_1, …c_n is exactly like d_1,…d_n if for every i, c_i is similar to d_i, and for every i, j, c_i=c_j iff d_i=d_j

1. As S is a subsentence of itself, it follows from Lemma 1 that M(S)=M*(S).
2. Therorem 1 is therefore proved.

Conclusion

- There is a decision procedure for monadic logic, but it has hyperexponential time complexity
- There exists monadic FOL proving systems:

  Prof. Manfred von Thun in La Trobe University, Australia did implement one with PASCAL.

  Example run:

  SOMEBODY IS rich AND SOMEBODY IS humble, EVERYBODY IS (rich OR humble), ALL rich PEOPLE ARE envied, ALL humble PEOPLE ARE friendly ----- EVERYBODY IS (envied AND friendly)?

  ... is not a valid argument, countermodel - rich = F:{ 2 3 } T:{ 1 } humble = F:{ 1 } T:{ 2 3 } envied = F:{ 3 } T:{ 1 } friendly = T:{ 2 3 }

- References:

Computability and Logic, George Boolos and Richard Jeffrey, Cambridge University Press, any edition

Modern Logic, Graeme Forbes, Oxford University Press, any edition