CAS 701 Fall 2004

02 What is Logic?

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What is Logic?

- Study of the principles of reasoning, particularly the notions of **valid inference** and **proof**.

- Branch of mathematics.

- Crucial distinction: the **syntax** versus the **semantics** of a language.

- Principal tools: formal systems called **logics**.

What is a Logic?

- Informally, a logic is a system of reasoning.

- Formally, a **logic** is a family of **formal languages** with:
  1. A common syntax.
  2. A common semantics.
  3. A notion of **logical consequence**.

- A logic may include a **proof system** for **proving** that a given formula is a logical consequence of a given set of formulas.

- Examples:
  - Propositional logic.
  - First-order logic.
  - Simple type theory (higher-order logic).

Language Syntax

- A language defines a collection of **expressions** formed from:
  - **Variables**.
  - **Constants** (nonlogical constants).
  - **Constructors** (logical constants).

- Two kinds of expressions:
  - **Terms**: Denote objects or values.
  - **Formulas**: Make assertions about objects or values.

- Some languages have constructors that bind variables (e.g., $\forall$, $\exists$, λ, I, ϵ, $\{ \mid \}$).

Language Semantics

- A **model** M for a language L is a pair (D, V) where:
  1. D is a set of values called the **domain** that includes the truth values t and f.
  2. V is a function from the expressions of L to D called the **valuation function**.

- M **satisfies** a formula A of L, written $M \models A$, if $V(A) = \mathrm{t}$.

- M **satisfies** a set Σ of formulas of L, written $M \models \Sigma$, if M satisfies each $A \in \Sigma$.

- Σ is **satisfiable** if there exists some model for L that satisfies Σ.

- A is **valid**, written $\models A$, if every model for L satisfies A.

- A is a **logical consequence** of Σ, written $\Sigma \models A$, if every model for L that satisfies Σ also satisfies A.

Hilbert-Style Proof System

- A **Hilbert-style proof system H** for a language L consists of:

  1. A set of formulas of L called **logical axioms**.
  2. A set of **rules of inference**.

- A **proof** of A from Σ in **H** is a finite sequence $B_1, \ldots, B_n$ of formulas of L with $B_n = A$ such that each B_i is either a logical axiom, a member of Σ, or follows from earlier B_j by one of the rules of inference.

- A is **provable** from Σ in **H**, written $\Sigma \vdash_{\mathbf{H}} A$, if there is a proof of A from Σ in **H**.

- A is a **theorem** in **H**, written $\vdash_{\mathbf{H}} A$, if A is provable from $\emptyset$ in **H**.

- Σ is **consistent** in **H** if not every formula is provable from Σ in **H**.

Kinds of Proof Systems

- Hilbert style.
- Symmetric sequent (Gentzen).
- Asymmetric sequent.
- Natural deduction (Quine, Fitch, Berry).
- Semantic tableaux (Beth, Hintikka).
- Resolution (J. Robinson).

Soundness and Completeness

- Let **P** be a proof system for a language L.

- **P** is **sound** if

$$\Sigma \vdash_{\mathbf{P}} A \quad \text{implies} \quad \Sigma \models A$$

- **P** is **complete** if

$$\Sigma \models A \quad \text{implies} \quad \Sigma \vdash_{\mathbf{P}} A$$

Theories

- A **theory** is a pair $T = (L, \Gamma)$ where:
  1. L is a language (the **language** of T).
  2. Γ is a set of formulas of L (the **axioms** of T).

- M is a **model** of T, written $M \models T$, if $M \models \Gamma$.

- A is **valid** in T, written $T \models A$, if $\Gamma \models A$.

- A is a **theorem** of T in **P**, written $T \vdash_{\mathbf{P}} A$, if $\Gamma \vdash_{\mathbf{P}} A$.

- T is **satisfiable** if Γ is satisfiable.

- T is **consistent** in **P** if Γ is consistent in **P**.