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03 Propositional Logic

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What is Propositional Logic?

- **Propositional logic** is the study of the truth or falsehood of **propositions** or **sentences** constructed using truth-functional **connectives**.
  - Also called **sentential logic**.

- Most other logics are extensions of propositional logic.

- Applications:
  - Logical arguments.
  - Switching networks.
  - Logical networks.

Syntax

- A **language** of proposition logic is a pair $L = (\mathcal{A}, \mathcal{B})$ where:
  - $\mathcal{A}$ is a set of constants called **propositional symbols** or **propositional letters**.
  - $\mathcal{B}$ is a set of 0-ary, unary, and binary constructors called **propositional connectives**.

- A **formula** of L is a string of symbols inductively defined by the following formation rules:
  1. Each $P \in \mathcal{A}$ is a formula of L.
  2. If $C_0, C_1, C_2 \in \mathcal{B}$ are 0-ary, unary, and binary, respectively, and A_1, A_2 are formulas of L, then C_0, $(C_1\ A_1)$, and $(A_1\ C_2\ A_2)$ are formulas of L.

- Common propositional connectives: T, F (0-ary); $\neg$ (unary); $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $|$ (binary).

An Example Language

- Let $L_0 = (\mathcal{A}, \mathcal{B})$ be the propositional language where:

  $\mathcal{A} = \{P_0, P_1, P_2, \ldots\}$.

  $\mathcal{B} = \{\neg, \Rightarrow\}$.

- The following abbreviations are employed:

T	denotes	$(P_0 \Rightarrow P_0)$.
F	denotes	$(\neg \mathsf{T})$.
$(A \vee B)$	denotes	$((\neg A) \Rightarrow B)$.
$(A \wedge B)$	denotes	$(\neg((\neg A) \vee (\neg B)))$.
$(A \Leftrightarrow B)$	denotes	$((A \Rightarrow B) \wedge (B \Rightarrow A))$.
$(A \mid B)$	denotes	$(\neg(A \wedge B))$.

Meaning of Propositional Connectives

- Each n-ary propositional connective is assigned a fixed n-ary truth function.

- Examples:

A	$(\neg A)$
t	f
f	t

A	B	$(A \Rightarrow B)$
t	t	t
t	f	f
f	t	t
f	f	t

A	B	$(A \Leftrightarrow B)$
t	t	t
t	f	f
f	t	f
f	f	t

A	B	$(A \wedge B)$
t	t	t
t	f	f
f	t	f
f	f	f

A	B	$(A \vee B)$
t	t	t
t	f	t
f	t	t
f	f	f

A	B	$(A \mid B)$
t	t	f
t	f	t
f	t	t
f	f	t

- A set $\mathcal{C}$ of proposition connectives is **complete** if every truth function can be represented by a formula using only members of $\mathcal{C}$. For example, $\{\neg, \Rightarrow\}$ and $\{|\}$ are complete.

Semantics

- Let $L = (\mathcal{A}, \mathcal{B})$ be a language of propositional logic, and for each $C \in \mathcal{B}$, let f_C be its assigned truth function.

- A **model** for L is an (interpretation) function I that assigns a truth value in $\{\mathsf{t}, \mathsf{f}\}$ to each $P \in \mathcal{A}$.

- The **valuation function** for I is the function V that maps formulas of L to $\{\mathsf{t}, \mathsf{f}\}$ and satisfies the following conditions:

  1. If $P \in \mathcal{A}$, then $V(P) = I(P)$.
  2. If $C \in \mathcal{B}$ is 0-ary, then $V(C) = f_C$.
  3. If $C \in \mathcal{B}$ is unary and A is a formula of L, then $V((C\ A)) = f_C(V(A))$.
  4. If $C \in \mathcal{B}$ is binary and A, B are formulas of L, then $V((A\ C\ B)) = f_C(V(A), V(B))$.

Truth Tables

- Let $L = (\mathcal{A}, \mathcal{B})$ be a language of propositional logic.

- A **tautology** of L is a valid formula of L, i.e., a formula that is assigned the value t in every model for L.

- Whether or not a formula of L is a tautology can be decided by the **method of truth tables**.
  - Hence propositional logic is **decidable**!

- Example (Rule of Contraposition):

A	B	$((A \Rightarrow B)$	$\Leftrightarrow$	$((\neg B)$	$\Rightarrow$	$(\neg A)))$
t	t	t	t	f	t	f
t	f	f	t	t	f	f
f	t	t	t	f	t	t
f	f	t	t	t	t	t

Laws of Propositional Logic

- The **laws of propositional logic** are fundamental laws of most other common logics.

- Examples:
  - Law of Double Negation.
  - Law of Excluded Middle.
  - Law of Contraposition.
  - De Morgan's Laws.
  - Associative, commutative, and distributive laws.
  - Idempotent laws.

A Hilbert-Style Proof System

Let **H** be the following Hilbert-style proof system for L_0:

- The **logical axioms** of **H** are all formulas of L_0 that are instances of the following three schemas:

  A1: $(A \Rightarrow (B \Rightarrow A))$.
  A2: $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$.
  A3: $((\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A))$.

- The single **rule of inference** of **H** is **modus ponens**:

  MP: From A and $(A \Rightarrow B)$, infer B.

Metatheorems of Propositional Logic

- **Deduction Theorem**. $\Sigma \cup \{A\} \vdash_{\mathbf{H}} B$ implies $\Sigma \vdash_{\mathbf{H}} A \Rightarrow B$.

- **Soundness Theorem**. $\Sigma \vdash_{\mathbf{H}} A$ implies $\Sigma \models A$.

- **Completeness Theorem**. $\Sigma \models A$ implies $\Sigma \vdash_{\mathbf{H}} A$.

- **Soundness and Completeness Theorem (second form)**. Σ is consistent in **H** iff Σ is satisfiable.

- **Compactness Theorem**. If Σ is finitely satisfiable, then Σ is satisfiable.