**CAS 701 Fall 2004**

# 04 Numbers, Sets, Functions, And Relations

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# Number Systems

- **N**, the natural numbers.
- **O**, the ordinal numbers.
- **Z**, the integers.
- $\mathbf{Z}_n$, the integers modulo $n$.
- **Q**, the rational numbers.
- **R**, the real numbers.
- **H**, the hyperreal numbers.
- **S**, the surreal numbers.
- **C**, the complex numbers.

# Foundational Mathematical Objects

- The three most common kinds of foundational objects:
  1. Sets.
  2. Functions.
  3. Relations.

- Each kind of object can be used to represent the other two kinds of objects.

# Sets

- A **set** is a collection of objects.

- Some very large collections of objects cannot be sets.
  - For example, consider the the **Russell set**, the set of all sets that do not contain themselves.

- Styles of set theories:
  - Naive set theory.
  - A universal set.
  - No universal set (e.g., **ZF set theory**).
  - A universal class (e.g., **NBG set theory**).

# Set Concepts

- Basic properties: membership, subset, cardinality.
- Basic operations:
  - Union, intersection, complement, difference, symmetric-difference.
  - Cartesian product (product), disjoint union (sum).
  - Power set.
- Special sets: the emptyset, universal sets, functions, relations, ordinals, cardinals.
- Functions and relations can be represented as special kinds of sets (e.g., as sets of **tuples**).

# Functions

- There are two definitions of a function:
  1. A **function** is a rule $f : I \to O$ that associates members of $I$ (inputs) with members of $O$ (outputs).
     - Every input is associated with at most one output.
     - Some inputs may not be associated with an output.
       Example: $f : \mathbf{Z} \to \mathbf{Q}$ where $x \mapsto 1/x$.
  2. A **function** is a set $F \subseteq I \times O$ such that if $(x, y), (x, y') \in F$, then $y = y'$.

- Each function $f$ has a **domain** $D \subseteq I$ and a **range** $R \subseteq O$.

- A set or relation can be represented as a special kind of function (e.g., as a **predicate**, a **characteristic function**, or an **indicator**).

# Function Concepts

- Basic properties:
  - arity (0, 1, $n \geq 2$, multiary).
  - total, injective, surjective, bijective.
  - image, reverse image.

- Basic operations: composition, restriction, inverse

- Special functions: the empty function, identity functions, choice functions

# Cardinality

- Two sets $A$ and $B$ are **equipollent**, written $A \approx B$, if there is a bijection $f : A \to B$ between them.

- $A \preceq B$ means $A \approx B'$ for some $B' \subseteq B$.

- A set is **infinite** if it is equipollent with a proper subset of itself.

- The **cardinality** of a set $A$ is the cardinal number $c$ such that $A$ and $c$ are equipollent.

- **Theorem**.

  1. $\mathbf{N} \approx \mathbf{Q}$.
  2. **(Cantor)** $\mathbf{N} \not\approx \mathbf{R}$.

- **Theorem (Schröder-Bernstein)**. If $A \preceq B$ and $B \preceq A$, then $A \approx B$.

# Relations

- An $n$-ary **relation** is a set $R \subseteq A_1 \times \cdots \times A_n$ ($n \geq 1$).
  - Any set can be considered as a unary relation.
  - Any nonunary relation can be considered as a binary relation.
- Functions are considered as special relations.
  - An $n$-ary function $f : A_1 \times \cdots \times A_n \to B$ is identified with the corresponding $(n+1)$-ary relation $R_f \subseteq A_1 \times \cdots \times A_n \times B$.
- An $n$-ary relation can be represented by an $n$-ary predicate.

# Relation Concepts

- Basic relation properties:
  - Reflexive, symmetric, transitive.
- Basic relation operations:
  - Domain, range.
  - Composition, inverse.
- Special relations: the empty relation, universal relations, equivalence relations.