CAS 701 Fall 2004

05 Mathematical Models

Instructor: W. M. Farmer

Revised: 5 October 2004

What is a Mathematical Model?

- A **mathematical model** is a representation of certain mathematical aspects of the world.

- There are two main kinds of mathematical models:
  - **Mathematical structures** consisting of a collection of sets, functions, and relations.
  - **Abstract machines** that perform computations.

- Mathematical models can either be
  - **Described** explicitly or
  - **Specified** by a set of properties.

Example: Program Specifications

Various kinds of functions and relations can used to specify computer programs:

- **Input/output specification**.
  - A function $f : I \rightarrow O$ that maps inputs to outputs.
  - A relation $R \subseteq I \times O$ that relates inputs and outputs.
- **Before/after specification**.
  - A function $f : I \times S \rightarrow O \times S$ that maps inputs and before-states to outputs and after-states.
  - A relation $R \subseteq I \times S \times O \times S$ that relates inputs, before-states, outputs, and after-states.
- **Trace specification**.
  - A function $f : I \times S^* \rightarrow O \times S^*$ that maps inputs and before-traces to outputs and after-traces.
  - A relation $R \subseteq I \times S^* \times O \times S^*$ relates inputs, before-traces, outputs, and after-traces.

Algebras

- An **algebra** is a mathematical structure consisting of:
  1. A set of elements called the **domain**.
  2. A set of distinguished elements, functions, and relations called the **signature** that impose a structure on the set of elements.

- The signature usually determines a **language** for describing and making assertions about the elements of the domain.

- Algebras are often described by tuples of the form

$$(D, e_1, \ldots, e_k, f_1, \ldots, f_m, r_1, \ldots .r_n).$$

Examples of Algebras

- Number systems (e.g, natural, ordinal, integer, rational, real, hyperreal, surreal, complex).
- Algebraic structures (e.g, monoids, groups, rings, fields).
- Orders (e.g., pre, partial, linear, well).
- Lattices and boolean algebras.
- Lists and sequences.
- Graphs and trees.
- Data structures used in Computer Science (e.g, arrays, stacks, queues).

Axiomatic Theories

- **Axiomatic theory** = formal language + set of axioms.

- Specifies a collection of mathematical structures (called the **models** of the theory).

- **Language**: vocabulary for describing and making assertions about the elements and their properties.

- **Axioms**: assumptions about the elements and properties.
  - Basis for proving theorems.

- Benefits:
  - **Conceptual clarity**: inessential details are omitted.
  - **Generality**: theorems hold in all models.

Abstract Machines

- An **abstract machine** or **automaton** is an abstract device for performing computations.

- A **universal machine** is a machine that can compute any computable function; examples:
  - Turing machine.
  - Unlimited register machine.

- A **state machine** is a machine that computes in a stepwise fashion in which each step includes receiving input, producing output, and changing state; examples:
  - Finite automaton.
  - Pushdown automaton.
  - Finite state machine.

Turing Machines (Alan Turing, 1936)

A **(deterministic) Turing machine** M consists of the following components:

1. A two-way infinite **tape** composed of cells.
2. A **tape head** that can read a cell and then (a) write a symbol in it, (b) move left, or (c) move right.
3. A **finite state control mechanism**.
4. Fixed finite set $\{s_1, \ldots, s_m\}$ of **tape symbols**.
5. Fixed finite set $\{q_1, \ldots, q_n\}$ of **states**.
6. A **transition relation** consisting of a finite set T of quadruples of the form (q_i, s_j, s_k, q_l), (q_i, s_j, L, q_l), or (q_i, s_j, R, q_l) such that (a) $1 \leq i, l \leq n$, (b) $0 \leq j, k \leq m$, and (c) for all (q_i, s_j), there is at most one $(q_i, s_j, \alpha, \beta) \in T$.

Finite State Machines

A **finite state machine** M consists of the following components:

1. A fixed finite set S of **states** including an **initial state**.

2. A fixed (possibly infinite) set I of **inputs**.

3. A fixed (possibly infinite) set O of **outputs**.

4. A **next state** relation ns $\subseteq S \times I \times S$.

5. An **output** relation out $\subseteq S \times I \times O$.

M is **deterministic** if the relations are functions, i.e., ns : $S \times I \to S$ and out : $S \times I \to O$.