CAS 701 Fall 2004

11 Simple Type Theory With Undefinedness

Instructor: W. M. Farmer

Revised: 15 November 2004

Undefinedness

- A mathematical term is **undefined** if it has no prescribed meaning or if it denotes a value that does not exist.

- Undefined terms are commonplace in mathematics, particularly in calculus.

- Sources of undefinedness:

  1. **Improper function applications**: $\sqrt{-4}$.
  2. **Improper definite descriptions**:

     "the x such that $x^2 = 4$".

  3. **Improper indefinite descriptions**:

     "some x such $x^2 = -4$".

Traditional Approach to Undefinedness

Based on three principles:

1. Atomic terms (i.e., variables and constants) are always defined.

2. Compound terms may be undefined.
   - A function application $f(a)$ is undefined if f is undefined, a is undefined, or $a \notin \mathrm{dom}(f)$.
   - A definite description "the x that has property P" is undefined if there is no x that has property P or there is more than one x that has property P.

3. Formulas are always true or false, and hence, are always defined.
   - A function application $p(a)$, where p is a predicate, is **false** if p is undefined, a is undefined, or $a \notin \mathrm{dom}(p)$.

Benefits of the Traditional Approach

- Meaningful statements can include undefined terms.

  $\forall\, x : \mathbf{R} \,.\, 0 \leq x \Rightarrow (\sqrt{x})^2 = x.$

  $0 \leq -2 \Rightarrow (\sqrt{-2})^2 = -2.$

- Function domains can be implicit.

  $k(x) = \frac{1}{x} + \frac{1}{x-1}.$

  $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$

- Definedness assumptions can be implicit.

  $\forall\, x, y, z : \mathbf{R} \,.\, \frac{x}{y} = z \Rightarrow x = y * z.$

- As a result, expressions involving undefinedness can be very concise.

Ways of Formalizing Undefinedness

1. Formalize partial functions and undefined terms in a standard logic using total functions and defined terms.
   - Several approaches.
   - Significant departure from the traditional approach which leads to verbose formal statements.

2. Formalize the traditional approach in a three-valued logic.
   - Flexible basis for formalizing undefinedness.
   - Significant departure from the traditional approach.

3. Formalize the traditional approach in a standard logic modified to admit undefined terms.
   - We call a logic of this kind a **logic with undefinedness**.
   - Traditional approach can be preserved.
   - Commits one to using a nonstandard logic.

Objective

Illustrate the third way of formalizing undefinedness by considering examples from calculus.

Steps:

1. Present STTwU, a version of simple type theory with undefinedness.

2. Observe how the traditional approach is used in calculus.

3. Formalize in STTwU examples from calculus involving partial functions and definite descriptions.

Which Calculus?

- We use M. Spivak's 1967 book *Calculus* for our investigation of undefinedness in calculus.

- *Calculus* is an outstanding calculus textbook.
  - Very rigorous, all theorems are proved.
  - Exceptionally careful about the issue of undefinedness, employs the traditional approach to undefinedness.
  - Has many interesting examples and exercises.
  - Introduces the student to mathematical practice and mathematical thinking.

- *Calculus* is an excellent place to see how undefinedness is handled in standard mathematical practice.

STTwU

- STT is a very simple variant of Church's type theory.
  - The primitive notions are function application, and function abstraction, equality, and definite description.
- STTwU is STT with undefinedness.
  - Syntax of STTwU: Exactly the same as STT's.
  - Semantics of STTwU: STT's semantics is modified to formalize the traditional approach to undefinedness.
  - STTwU is one of several standard logics with undefinedness that have been proposed.
- There is a proof system for STTwU that is complete with respect to the general models semantics for STTwU.
- STTwU is a simple version of the logic of the IMPS theorem proving system.
  - Thus, whether STTwU can be effectively implemented is not an issue.

Syntax of STTwU: Types

- A **type** of STTwU is defined by the following rules:

$$\textbf{T1}\ \frac{}{\textbf{type}[\iota]}\ \textbf{(Type of individuals)}$$

$$\textbf{T2}\ \frac{}{\textbf{type}[*]}\ \textbf{(Type of truth values)}$$

$$\textbf{T3}\ \frac{\textbf{type}[\alpha],\ \textbf{type}[\beta]}{\textbf{type}[(\alpha \to \beta)]}\ \textbf{(Function type)}$$

- Let $\mathcal{T}$ denote the set of types of STTwU.

Syntax of STTwU: Symbols

- The **logical symbols** of STTwU are:
  - **Function application**: `@` (hidden).
  - **Function abstraction**: λ.
  - **Equality**: $=$.
  - **Definite description**: `I` (capital iota).
  - An infinite set $\mathcal{V}$ of symbols called **variables**.

- A **language** of STTwU is a pair $L = (\mathcal{C}, \tau)$ where:
  - $\mathcal{C}$ is a set of symbols called **constants**.
  - $\tau : \mathcal{C} \to \mathcal{T}$ is a total function.

Syntax of STTwU: Expressions

- An **expression** E of **type** α of a STTwU language $L = (\mathcal{C}, \tau)$ is defined by the following rules:

E1 $$\frac{x \in \mathcal{V},\ \mathbf{type}[\alpha]}{\mathbf{expr}_L[(x : \alpha), \alpha]}$$ **(Variable)**

E2 $$\frac{c \in \mathcal{C}}{\mathbf{expr}_L[c, \tau(c)]}$$ **(Constant)**

E3 $$\frac{\mathbf{expr}_L[A, \alpha],\ \mathbf{expr}_L[F, (\alpha \to \beta)]}{\mathbf{expr}_L[F(A), \beta]}$$ **(Application)**

E4 $$\frac{x \in \mathcal{V},\ \mathbf{type}[\alpha],\ \mathbf{expr}_L[B, \beta]}{\mathbf{expr}_L[(\lambda\, x : \alpha\ .\ B), (\alpha \to \beta)]}$$ **(Abstraction)**

E5 $$\frac{\mathbf{expr}_L[E_1, \alpha],\ \mathbf{expr}_L[E_2, \alpha]}{\mathbf{expr}_L[(E_1 = E_2), *]}$$ **(Equality)**

E6 $$\frac{x \in \mathcal{V},\ \mathbf{type}[\alpha],\ \mathbf{expr}_L[A, *]}{\mathbf{expr}_L[(\mathrm{I}\, x : \alpha\ .\ A), \alpha]}$$ **(Definite description)**

Semantics of STTwU

The semantics of STTwU is the same as the semantics of STT except that:

1. A model contains **partial and total functions** instead of just total functions.

2. The value of an improper function application is **false** if it is a formula and is **undefined** if it is not a formula.

3. The value of a function abstraction is a function that is **possibly partial**.

4. The value of an equality is **false** if the value of either of its arguments is undefined.

5. The value of an improper definite description is **false** if it is a formula and is **undefined** if it is not a formula.

Definitions and Abbreviations

T	means	$(\lambda\, x : * \,.\, x) = (\lambda\, x : * \,.\, x)$.
F	means	$(\lambda\, x : * \,.\, \mathsf{T}) = (\lambda\, x : * \,.\, x)$.
$(\neg A_*)$	means	$A_* = \mathsf{F}$.
$(A_\alpha \neq B_\alpha)$	means	$\neg(A_\alpha = B_\alpha)$.
$(A_* \wedge B_*)$	means	$(\lambda\, f : * \to (* \to *) \,.\, f(\mathsf{T})(\mathsf{T})) =$ $(\lambda\, f : * \to (* \to *) \,.\, f(A_*)(B_*))$.
$(A_* \vee B_*)$	means	$\neg(\neg A_* \wedge \neg B_*)$.
$(A_* \Rightarrow B_*)$	means	$\neg A_* \vee B_*$.
$(\forall\, x : \alpha \,.\, A_*)$	means	$(\lambda\, x : \alpha \,.\, A_*) = (\lambda\, x : \alpha \,.\, \mathsf{T})$.
$(\exists\, x : \alpha \,.\, A_*)$	means	$\neg(\forall\, x : \alpha \,.\, \neg A_*)$.
$(A_\alpha\!\downarrow)$	means	$\exists\, x : \alpha \,.\, x = A_\alpha$.
$(A_\alpha\!\uparrow)$	means	$\neg(A_\alpha\!\downarrow)$.
$(A_\alpha \simeq B_\alpha)$	means	$(A_\alpha\!\downarrow \vee B_\alpha\!\downarrow) \Rightarrow A_\alpha = B_\alpha$.
$\bot_\alpha$	means	$\mathsf{I}\, x : \alpha \,.\, x \neq x$.
$\mathsf{if}(A_*, B_\alpha, C_\alpha)$	means	$\mathsf{I}\, x : \alpha \,.\, (A_* \Rightarrow x = B_\alpha) \wedge (\neg A_* \Rightarrow x = C_\alpha)$ where x does not occur in A_*, B_α, or C_α.

The Properties of the Real Numbers

- Spivak's development of calculus begins with 13 basic properties of the real numbers—which are essentially the axioms of a **complete ordered field**.

- We construct a theory named COF in STTwU that is an extremely direct formalization of Spivak's 13 properties.

- The only significant difference between COF and Spivak's 13 properties is that COF also includes the axiom.

  $$0^{-1}\uparrow$$

  which Spivak only states informally.

Examples: Partial Functions

- Example 2:

  Spivak: $k(x) = \frac{1}{x} + \frac{1}{x-1}$ [p. 39].

  STTwU: $\forall\, x\, .\, k(x) \simeq \frac{1}{x} + \frac{1}{x-1}$.

- Example 3:

  Spivak: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ [p. 41].

  STTwU : $\forall\, f, g, x$: fun_div$(f)(g)(x) \simeq \frac{f(x)}{g(x)}$

  where fun_div : $((\iota \to \iota) \to ((\iota \to \iota) \to (\iota \to \iota)))$.

Examples: Limits (1)

- Example 4:

  Spivak: $\lim_{x \to a} f(x)$ denotes the real number l such that, for every $\epsilon > 0$, there is some $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$, then $|f(x) - l| < \epsilon$
  [pp. 78, 81].

  STTwU: $\forall\, f, a \,.\, \mathsf{lim}(f)(a) \simeq$
  $(\mathrm{I}\, l \,.$
  $(\forall\, \epsilon \,.\, 0 < \epsilon \Rightarrow$
  $(\exists\, \delta \,.\, 0 < \delta \wedge$
  $(\forall\, x \,.\, (0 < \mathsf{abs}(x - a) \wedge \mathsf{abs}(x - a) < \delta) \Rightarrow$
  $\mathsf{abs}(f(x) - l) < \epsilon))))$.

Example: Limits (2)

- Example 5:

  Spivak: If $\lim_{x \to a} g(x) = m$ and $m \neq 0$, then $\lim_{x \to a} \left(\frac{1}{g}\right)(x) = \frac{1}{m}$ [Theorem 2(3), p. 84].

  STTwU: $\forall\, g, a, m\ .$
  $(\lim(g)(a) = m \wedge m \neq 0) \Rightarrow \lim(\text{fun\_div}(\text{id})(g))(a) = \frac{1}{m}$.

- Example 6:

  Spivak: The function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$ [p. 93].

  STTwU: $\forall\, f, a\ .\ \text{cont}(f)(a) \Leftrightarrow \lim(f)(a) = f(a)$.

Conclusion

- We have shown that the traditional approach to undefinedness can be directly formalized in a traditional logic with undefinedness.
  - In particular, the conciseness that comes from the use the traditional approach can be fully preserved.

- We recommend that logics with undefinedness, such as STTwU, be considered as the logical basic for mechanized mathematics systems.
  - They are closer to mathematical practice with respect to undefinedness than standard logics.
  - They can be effectively implemented as demonstrated by IMPS.