CAS 701 Fall 2002

Final Examination

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You have 3 hours to complete this test worth 100 points consisting of 3 pages and 8 questions. Write your answers in the examination booklet provided to you. Good luck!

(1) [10 pts.] Use truth tables to verify that the de Morgan Laws for propositional logic are tautologies:

   (a) $\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$.
   (b) $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$.

(2) [10 pts.] Find a set S of real numbers such that the order type of $(S, \leq)$ is γ where:

   (a) $\gamma = \omega$.
   (b) $\gamma = \omega + \omega$.
   (c) $\gamma = \omega \cdot \omega$.

(3) [20 pts.] Let A be a set with $|A| > 1$, F be the set of all total functions $f : A \rightarrow A$, and $\circ : F \times F \rightarrow F$ be composition (i.e., for all $f, g \in F$ and $x \in A$, $(f \circ g)(x) = f(g(x))$).

   (a) Show that there is some $h \in F$ such that $(F, \circ, h)$ is a monoid.
   (b) Show that there is some F' with $\{h\} \subset F' \subset F$ and some $i : F \rightarrow F$ such that $(F', \circ, i, h)$ is a group.

(4) [10 pts.] Assume that the following statements hold:

   - F is a sound and complete formal system for a language L.
   - $T = (L, \Gamma)$ is a theory such that T is consistent in F, T is complete, and Γ is finite.

   Describe a decision procedure for T, i.e., describe an algorithm that, given a formula φ of L, determines whether $T \models \varphi$ or $T \models \neg\varphi$. Explain why your procedure is correct.

Test continues on next page.

(5) [10 pts.] Consider the theory $T = (L, \Gamma)$ of first-order logic such that:

$L = (\mathcal{V}, \emptyset, \emptyset, \{=, <\})$ where $=$ and $<$ are binary.

$\Gamma = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5\}$ where:

φ_1 is $\forall\, x \,.\, \neg(x < x)$.

φ_2 is $\forall\, x, y, z \,.\, x < y \wedge y < z \Rightarrow x < z$.

φ_3 is $\forall\, x, y \,.\, x = y \vee x < y \vee y < x$.

φ_4 is $\forall\, x, y \,.\, \exists z \,.\, x < y \Rightarrow x < z \wedge z < y$.

φ_5 is $\exists\, x, y \,.\, \neg(x = y)$.

(a) Describe four nonisomorphic models of T of cardinality $\leq \omega$.

(b) Describe one uncountable model of T of cardinality $> \omega$.

(6) [10 pts.] Formalize in simple type theory (STT) the theory of 3-colorable graphs. A graph is *3-colorable* if the nodes of the graph can be colored using three different colors such that, for all nodes n_1, n_2 in the graph, if there is an edge from n_1 to n_2, then n_1 and n_2 have different colors.

(7) [10 pts.] Recall that the *Ackermann function* $A : \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$ is defined recursively by the following equations:

$$A(0, y) = y + 1.$$
$$A(x + 1, 0) = A(x, 1).$$
$$A(x + 1, y + 1) = A(x, A(x + 1, y)).$$

Explain how this definition can be viewed as an instance of well-founded recursion. Hint: The well-founded relation is a certain lexicographical order.

(8) [20 pts.] Consider the theory $T = (L, \Gamma)$ of simple type theory:

$$L = (\mathcal{V}, \{A, P, V, L, R\}, \tau) \text{ where}$$

$$\tau(A) = (* \to \iota).$$
$$\tau(P) = (\iota \to (\iota \to \iota)).$$
$$\tau(V) = (\iota \to *).$$
$$\tau(L) = (\iota \to \iota).$$
$$\tau(R) = (\iota \to \iota).$$

$\Gamma = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7\}$ where:

φ_1 is $\forall\, x : * \,.\, \forall\, s, t : \iota \,.\, \neg(A(x) = P(s, t))$.

φ_2 is $\forall\, x_1, x_2 : * \,.\, A(x_1) = A(x_2) \Rightarrow x_1 = x_2$.

φ_3 is $\forall\, s_1, s_2, t_1, t_2 : \iota \,.$
$P(s_1, t_1) = P(s_2, t_2) \Rightarrow (s_1 = s_2 \wedge t_1 = t_2)$.

φ_4 is $\forall\, x : * \,.\, V(A(x)) = x$.

φ_5 is $\forall\, s, t : \iota \,.\, L(P(s, t)) = s$.

φ_6 is $\forall\, s, t : \iota \,.\, R(P(s, t)) = t$.

φ_7 is $\forall\, Q : (\iota \to *) \,.$
$[(\forall\, x : * \,.\, Q(A(x))) \wedge$
$(\forall\, s, t : \iota \,.\, (Q(s) \wedge Q(t)) \Rightarrow Q(P(s, t)))] \Rightarrow$
$\forall\, t : \iota \,.\, Q(t)$.

T is a theory of binary trees whose leaves are truth values.

(a) Which of the axioms of T ensure that there is "no confusion" in the models of T?

(b) Which of the axioms of T ensure that there is "no junk" in the models of T?

(c) By recursion, define $M : (\iota \to \iota)$ in T to be the function that, given a tree t, returns the tree t' that is the "mirror" of t. Your definition should have the form

$$M = (\lambda\, t : \iota \,.\, B).$$

(d) Explain why your definition of M makes sense.

(e) Prove in T that

$$\forall\, t : \iota \,.\, M(M(t)) = t.$$

Test ends here.