Computing and Software 701

Logic and Discrete Mathematics
In Software Engineering

Fall 2004

Final Exam

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You have 3 hours to complete this exam consisting of 2 pages and 6 questions. Write your answers in the examination booklet provided to you. Give reasons for your answers. Good luck!

(1) Let $f : A \to B$ and $g : B \to C$ be total, and let $h = g \circ f : A \to C$ be the composition of g and f.

  (a) [5 pts.] If h is injective, does it follows that f is injective? Give a proof or counterexample.

  (b) [5 pts.] If h is injective, does it follows that g is injective? Give a proof or counterexample.

(2) [10 pts.] Let $L = (\emptyset, \emptyset, \{=, <\})$ be a language of FOL. Find a set Γ of formulas of L such that $T = (L, \Gamma)$ is a first-order theory of strict partial orders that do not have a maximum element.

(3) [10 pts.] Let T be the full infinite (ω) branching tree of infinite (ω) height. (The branches at each node are labeled $0, 1, 2, \ldots$.) What is the cardinality of the set of infinite paths in T that start at the root of the tree? You may use results proved in class or as exercises.

(4) [10 pts.] Consider the algebra $A = (S, \mathsf{cat})$ where S is the set of strings over the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots, \mathtt{x}, \mathtt{y}, \mathtt{z}\}$, and cat is string concatenation. Show that A has the structure of a monoid.

(5) [10 pts.] Let $\mathsf{COF} = (L, \Gamma)$ be the theory of a complete ordered field defined in class. Using function abstraction and definite description, find an expression E whose value in a model of COF is the division function.

(6) Let $\mathsf{COF}' = (L, \Gamma)$, where $L = (\{\mathsf{Real}\}, \mathcal{C}, \tau)$, be the theory of a complete ordered field in many-sorted STT that is obtained from COF by renaming ι to Real. Let $T = (L', \Gamma \cup \Gamma')$ where:

Test continues on next page.

$L' = (\{\mathsf{Real}, \mathsf{List}\}, \mathcal{C} \cup \mathcal{C}', \tau')$, $\mathcal{C}' = \{\mathsf{nil}, \mathsf{cons}, \mathsf{head}, \mathsf{tail}\}$, τ is a subfunction of τ', and τ' on $\mathcal{C}'$ is defined by the following table:

Constant c	**Type** $\tau'(c)$
nil	List
cons	Real → (List → List)
head	List → Real
tail	List → List

Γ' contains the following formulas of L':

1. $\forall\, r : \mathsf{Real}, x : \mathsf{List} \,.\, \mathsf{nil} \neq \mathsf{cons}(r)(x)$.
2. $\forall\, r_1, r_2 : \mathsf{Real}, x_1, x_2 : \mathsf{List} \,.$
   $\mathsf{cons}(r_1)(x_1) = \mathsf{cons}(r_2)(x_2) \Rightarrow (r_1 = r_2 \wedge x_1 = x_2)$.
3. $\forall\, r : \mathsf{Real}, x : \mathsf{List} \,.\, \mathsf{head}(\mathsf{cons}(r)(x)) = r$.
4. $\forall\, r : \mathsf{Real}, x : \mathsf{List} \,.\, \mathsf{tail}(\mathsf{cons}(r)(x)) = x$.
5. $\forall\, p : \mathsf{List} \to * \,.$
   $(p(\mathsf{nil}) \wedge (\forall\, x : \mathsf{List} \,.\, p(x) \Rightarrow \forall\, r : \mathsf{Real} \,.\, p(\mathsf{cons}(r)(x))))$
   $\Rightarrow \forall\, x : \mathsf{List} \,.\, p(x)$.

T is a theory of (finite) lists of real numbers such that

$$\mathsf{cons}(2)(\mathsf{cons}(17)(\mathsf{cons}(31)(\mathsf{nil})))$$

represents the list $\langle 2, 17, 31 \rangle$. The *sum* of this list is the real number $2 + 17 + 31$. The *product* of a real number r and this list is the list $\langle r * 2, r * 17, r * 31 \rangle$.

(a) [5 pts.] Which of the axioms in Γ' ensure that there is "no confusion" in the models of T?

(b) [5 pts.] Which of the axioms in Γ' ensure that there is "no junk" in the models of T?

(c) [10 pts.] Define by well-founded recursion a constant sum of type List → Real to be the function that gives the sum of a list. What is the well-founded relation?

(d) [10 pts.] Write a term rewrite system $\mathcal{R}$ that computes the sum of a ground expression of type List and that is sound with respect to T, confluent, and finitely terminating (but not complete with respect to T).

(e) [10 pts.] Define by well-founded recursion a constant prod of type Real → (List → List) to be the function that gives the product of a real number and a list. What is the well-founded relation?

(f) [10 pts.] Using axiom 5, prove

$$T \models \forall r : \mathsf{Real}, x : \mathsf{List} \,.\, \mathsf{sum}(\mathsf{prod}(r)(x)) = r * \mathsf{sum}(x).$$

Test ends here.