CAS 701 Fall 2002

Midterm Test

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You have 110 minutes to complete this test consisting of 2 pages and 8 questions. Write your answers in the examination booklet provided to you. Good luck!

(1) Let A_1, A_2, B_1, B_2 be nonempty sets such that $A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$. Prove the statement below that is true.

  (a) $[(A_1 \times B_1) \cup (A_2 \times B_2)] \subset [(A_1 \cup A_2) \times (B_1 \cup B_2)]$.
  (b) $[(A_1 \times B_1) \cup (A_2 \times B_2)] = [(A_1 \cup A_2) \times (B_1 \cup B_2)]$.
  (c) $[(A_1 \times B_1) \cup (A_2 \times B_2)] \supset [(A_1 \cup A_2) \times (B_1 \cup B_2)]$.

(2) Let $f : \mathbf{N} \rightarrow \mathbf{N}$ be the total function $\lambda\, x \,.\, x + 1$ (i.e., $\forall\, x \in \mathbf{N} \,.\, f(x) = x + 1$). For $n \geq 1$, define

$$f_n = \begin{cases} f & \text{if } n = 1 \\ f \circ f_{n+1} & \text{if } n > 1 \end{cases}$$

and R_n to be the range of f_n.

  (a) For $n \geq 1$, what is the cardinality of R_n?
  (b) What is the cardinality of

$$\bigcap_{n \geq 1} R_n.$$

(3) Let the *symmetric closure* of $R \subseteq A \times A$ be the least $R' \subseteq A \times A$ such that R' is symmetric and $R \subseteq R'$. Given $R \subseteq A \times A$, write an expression for the symmetric closure of R that shows how it is constructed from R.

(4) Recall that $(\mathbf{N}, +, 0)$ is a monoid.

  (a) Let $\mathbf{E}$ be the set of even natural numbers. Show that $(\mathbf{E}, +, 0)$ is a monoid.
  (b) Let $\mathbf{O}$ be the set of odd natural numbers. Show that $(\mathbf{O}, +, 1)$ is not a monoid.

Test continues on next page.

(5) Let $(\mathbf{B}, <_{\mathbf{B}})$ and $(\mathbf{N}, <_{\mathbf{N}})$ be (strict) well-orders such that $\mathbf{B}$ is the set $\{\text{T}, \text{F}\}$ of truth values and $\mathbf{N}$ is the set of natural numbers. Define $W_1 = (\mathbf{B} \times \mathbf{N}, <_1)$ to be the well-order where $\mathbf{B} \times \mathbf{N}$ is ordered lexicographically by $<_1$ (i.e., the formula

$$\forall\, t, t' \in \mathbf{B}, n, n' \in \mathbf{N} \;.\; (t, n) <_1 (t', n') \Leftrightarrow t <_{\mathbf{B}} t' \vee (t = t \wedge n <_{\mathbf{N}} n')$$

holds). Define $W_2 = (\mathbf{N} \times \mathbf{B}, <_2)$ in a similar way.

(a) What is the cardinality of $\mathbf{B} \times \mathbf{N}$?

(b) What is the cardinality of $\mathbf{N} \times \mathbf{B}$?

(c) What ordinal has the same order type as W_1?

(d) What ordinal has the same order type as W_2?

(6) Let F be a sound formal system for a language L of First-Order Logic (FOL). Suppose the theory $T = (L, \Gamma)$ of FOL is *inconsistent* in F. Prove that T is *unsatisfiable*.

(7) Let $L = \{P, Q, R\}$ be a language of Propositional Logic (PL), and let φ be the following formula of L:

$$[(P \wedge Q) \Rightarrow R] \Rightarrow [\neg R \Rightarrow (P \vee Q)].$$

(a) How many models are there for L?

(b) Is φ a tautology, i.e., is φ a valid formula of L?

(c) How many models for L satisfy φ?

(d) If T is the theory $(L, \{P \Leftrightarrow \mathsf{T}\})$, where T is the logical constant that denotes the true value, how many models for L are models of T?

(8) Let $L = (\mathcal{V}, \{a, b, c\}, \emptyset, \emptyset)$ be a language of First-Order Logic (FOL) and $T = (L, \{\varphi\})$ be a theory of FOL where φ is

$$\forall\, x \;.\; x = a \vee x = b \vee x = c.$$

Suppose $M = (D, I)$ is a model for L with $D = D' \cup \{\text{T}, \text{F}\}$ and $D' \cap \{\text{T}, \text{F}\} = \emptyset$. What is the cardinality of D' if $M \models T$.

Test ends here.