Computing and Software 701

Logic and Discrete Mathematics In Software Engineering

Fall 2008

Midterm Test Answer Key

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You have 110 minutes to complete this test consisting of 2 pages and 6 questions. Write your answers in the examination booklet provided to you. Give reasons for your answers. The use of any calculators, notes, and books is permitted during this exam, but you may not use any other electronic devices. Good luck!

(1) [5 pts.] Explain how a satisfiability checker for a traditional logic like propositional logic or FOL can be used to check the validity of the formulas of the logic.

Answer: A formula in a traditional logic is satisfiable iff its negation is not valid. Therefore, a satisfiability checker can be used to check whether a formula A is valid as follows. The satisfiability checker is applied to $\neg A$. If true is returned, then A is not valid, and if false is returned, then A is valid.

(2) Suppose A is a propositional formula with the following truth table:

p	q	r	A
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

(a) [10 pts.] Construct a propositional formula B in conjunctive normal form such that A and B are logically equivalent.

Answer:$(\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r)$.

(b) [10 pts.] Construct a propositional formula B in disjunctive normal form such that A and B are logically equivalent.

Answer:
$(p \wedge q \wedge r) \vee$
$(p \wedge \neg q \wedge \neg r) \vee$
$(\neg p \wedge q \wedge r) \vee$
$(\neg p \wedge q \wedge \neg r) \vee$
$(\neg p \wedge \neg q \wedge r) \vee$
$(\neg p \wedge \neg q \wedge \neg r)$.

(3) [10 pts.] Let R_1 and R_2 be equivalence relations on a set S. Prove that $R_1 \cup R_2$ is also an equivalence relation on S or give a counterexample in which $R_1 \cup R_2$ is not an equivalence relation.

Answer: This claim is false as shown by the following counterexample. Let $S = \{a, b, c\}$, $R_1 = \{(a,a),(a,b),(b,a),(b,b),(c,c)\}$, and $R_1 = \{(a,a),(b,b),(b,c),(c,b),(c,c)\}$. Clearly, both R_1 and R_2 are equivalence relations on S. Assume $R = R_1 \cup R_2$ is transitive. Then $a\,R\,b$ and $b\,R\,c$ implies $a\,R\,c$, which is false. Hence R is not transitive and not an equivalence relation.

(4) Suppose F is the set of partial functions $f : \mathbf{N} \to \mathbf{N}$ where $\mathbf{N}$ denotes the set of natural numbers. (Recall that a partial function is a function $f : A \to B$ that is undefined on some members of A.) For $f, g \in F$, f is a *subfunction* of g, written $f \sqsubseteq g$, if the domain D_f of f is a subset of the domain of g and, for all $x \in D_f$, $f(x) = g(x)$.

(a) [10 pts.] Show that $(F, \sqsubseteq)$ is a weak partial order.

Answer: We must show that $(F, \sqsubseteq)$ is reflexive, antisymmetric, and transitive.

i. Obviously, for all $f \in F$, $f \sqsubseteq f$, so $(F, \sqsubseteq)$ is reflexive.
ii. Suppose $f \sqsubseteq g$ and $f \sqsubseteq g$. Then $D_f \subseteq D_g$ and $D_g \subseteq D_f$, which implies $D_f = D_g$ and so $f = g$. Hence $(F, \sqsubseteq)$ is antisymmetric.
iii. Suppose $f \sqsubseteq g$ and $g \sqsubseteq h$. Then $D_f \subseteq D_g$ and $D_g \subseteq D_h$, which implies $D_f \subseteq D_h$. Also, for all $x \in D_f$, $f(x) = g(x)$ and, for all $x \in D_g$, $g(x) = h(x)$. This implies, for all $x \in D_f$, $f(x) = g(x) = h(x)$, and so $f \sqsubseteq h$. Hence $(F, \sqsubseteq)$ is transitive.

Therefore, $(F, \sqsubseteq)$ is a weak partial order.

(b) [5 pts.] Describe the set of minimal elements of F.

Answer: The set of minimal elements of F is the singleton set containing the empty function (i.e., the function whose domain is empty).

(c) [5 pts.] Describe the set of maximal elements of F.

Answer: The set of maximal elements of F is the set of functions in F that are undefined on exactly one member of $\mathbf{N}$.

(d) [5 pts.] Does F have a minimum element? If so, what is it?

Answer: Yes, the empty function is the minimum element of F.

(e) [5 pts.] Does F have a maximum element? If so, what is it?

Answer: No, F has no maximum element since it has more than one maximal element.

(f) [10 pts.] Show that $(F, \sqsubseteq)$ is a meet-semilattice.

Answer: We must show that, for all $f, g \in F$, the greatest lower bound of $\{f, g\}$ exists. Let $h \in F$ be defined as follows. For all $n \in \mathbf{N}$, if $f(n) = g(n)$, then $h(n) = f(n)$, and otherwise $h(n)$ is undefined. h is clearly a lower bound of $\{f, g\}$. Suppose h' is a lower bound of $\{f, g\}$. If $n \in D_{h'}$, then $h'(n) = f(n)$ and $h'(n) = g(n)$, and so $h'(n) = h(n)$. Hence, $h' \sqsubseteq h$, and so h is the greatest lower bound of $\{f, g\}$. Therefore, $(F, \sqsubseteq)$ is a meet-semilattice.

(5) [10 pts.] Suppose FOL is extended to a version of first-order logic called FOL$'$ such that, if B is a formula of FOL$'$, then

$$\exists !\, x \,.\, B$$

is a formula of FOL$'$. $\exists !\, x \,.\, B$ is intended to mean "there is a unique x that satisfies B". FOL and FOL$'$ have the same set of languages. Given a model $M = (D, I)$ for a language of FOL$'$ and variable assignment φ into M, define $V_\varphi^M(A)$ when $A = \exists !\, x \,.\, B$.

Answer: If $V_{\varphi[x \mapsto d]}^M(B) = \textsc{t}$ for exactly one $d \in D$, then $V_\varphi^M(A) = \textsc{t}$; otherwise $V_\varphi^M(A) = \textsc{f}$.

(6) Let $\mathbf{PA}' = (L', \Gamma')$ with $L' = (\{0,1\}, \{S, +, *\}, \{=\})$ be the formulation of first-order Peano arithmetic presented in class. Let $\mathbf{PA}'' = (L'', \Gamma'')$, where $L'' = (\{0,1\}, \{S, f, +, *\}, \{=, <\})$ and $\Gamma'' = \Gamma' \cup \{A_1, A_2, A_3\}$, be a theory that extends $\mathbf{PA}'$. f is a unary function symbol and $<$ is a binary predicate symbol.

(a) [5 pts.] A_1 is a sentence that defines $<$ to be the usual strict linear order on the natural numbers. Write a sentence of L'' that says this.

Answer: $\forall\, m, n \,.\, m < n \Leftrightarrow (\exists p \,.\, \neg(p = 0) \wedge m + p = n)$.

(b) [5 pts.] A_2 is a sentence that asserts that f has a fixed point. Write a sentence of L'' that says this.

Answer: $\exists\, m \,.\, f(m) = m$.

(c) [5 pts.] A_3 is a sentence that asserts that f has at most finitely many fixed points. Write a sentence of L'' that says this.

Answer: $\exists\, n \,.\, \forall\, m \,.\, (f(m) = m \Rightarrow m < n)$.