Solutions of Exercise Group Number 1
Logic and Discrete Mathematics in Software Engineering
Fall 2008

Exercise 2

- Valid: b and f.
- Not Valid: a, c, d and f.
- Satisfiable: b, c, d, e, f.
- Not satisfiable: a

Exercise 3 (Knights and Knaves).

We use the following propositions:

- A: A is a Knight.
- B: B is a Knight.

then the idea is to formalize the restrictions of our system and then find truth assignments that satisfy our restrictions. We know: *A says: two of us are both knights* and *B says: A is a knave*, also we know that Knights always tell the true and Knaves always tell lies. Therefore we can establish:

- $A \Rightarrow (A \wedge B)$ (if A is knight then A and B are knights)
- $(\neg A) \Rightarrow \neg(A \wedge B)$ (if A is Knave then he lies)
- $B \Rightarrow (\neg A)$ (if B is Knight then A is a Knave)
- $(\neg B) \Rightarrow A$ (if B is a Knave then A is a Knight)

We have to find a truth assignment that satisfy these formulas:

A	B	$A \Rightarrow (A \wedge B)$	$(\neg A) \Rightarrow \neg(A \wedge B)$	$B \Rightarrow (\neg A)$	$(\neg B) \Rightarrow A$
t	t	t	t	f	t
f	f	t	t	t	f
f	t	t	t	t	t
t	f	f	t	t	t

That is, the only assignment that satisfies this set of formulas is: $A\ is\ false$ and $B\ is\ true$, therefore:

- A is Knave.
- B is a Knight.

b)

We know:

- A says *I am a Knight*
- B says *I am a Knight*

Then the following formulae must be true:

- $A \Rightarrow A$ (if A is knight then he is a Knight) **tautology**
- $(\neg A) \rightarrow (\neg A)$ (if A is a Knave then he is a Knave) **tautology**
- $B \Rightarrow B$ (if B is a Knight then he is a Knight) **tautology**
- $(\neg B) \Rightarrow (\neg B)$ (if B is a Knave then he is a Knave) **tautology**

Since all are tautologies every truth assignment satisfies the restrictions, this means that all the combinations are possible:

- A is a Knight and B is a Knave.
- B is Knight and A is a Knave.
- A is a Knave and B is a Knave.
- A is a Knight and B is a Knave.

Exercise 4

Producing the truth table we find that $A \mid (B \mid C)$ is not equivalent to $(A \mid B) \mid C$.

Exercise 5

Having that $\{\neg, \vee\}$ are a complete set of connectives, we show with truth tables that the following equivalences are tautologies

1. $(\neg p) \Leftrightarrow (p \mid p)$
2. $(p \vee q) \Leftrightarrow ((p \mid q) \mid (p \mid q))$

That is, we can use 1 and 2 to rewrite every formula with logical constants $\neg, \vee$ to formula with only the logic constant $|$. And therefore $\{|\}$ is a complete set of connectives.

Exercise 6

- Definition of *Conjunctive Normal Form*: An expression is in **Conjunctive Normal Form** if and only if it is a conjunction consisting of one of more conjuncts, each of which is a disjunction consisting of propositional symbols or negations of propositional symbols. For example: A, $(A \vee B) \wedge (\neg A \vee \neg B)$ are in Conjunctive Normal form.

- Definition of *Disjunctive Normal Form*: An expression is in **Disjunctive Normal Form** if and only if it is a disjunction consisting of one of more disjuncts, each of which is conjunction consisting of propositional symbols, or negations of propositional symbols. For example: $(A \wedge B) \vee (\neg A \wedge C) \vee (\neg C)$, $(A \wedge B) \vee (A \wedge C)$ are in disjunctive normal form.

Exercise 7

a)

$(\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q) \wedge (\neg p \vee \neg r) \wedge \neg q \wedge \neg r$

b)

$(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$

Exercise 8

To translate a formula with connectives $\{\Rightarrow, \wedge, \vee, \neg, \Leftrightarrow\}$ to conjunctive normal form will use the following equivalences:

1. $(p \Leftrightarrow q) \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
2. $(p \Rightarrow q) \equiv (\neg p \vee q)$

3. $\neg(\neg p) \equiv p$

4. $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$

5. $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$

6. $r \vee (p \wedge q) \equiv (r \vee p) \wedge (r \vee q)$

7. $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$

Given a formulae ϕ we can use the following procedure to translate it to a Conjunctive Normal Form:

Step 1. Use equivalences 1,2 to remove all symbols $\Rightarrow, \Leftrightarrow$ from the formulae.

Step 2. Use equivalences 3,4,5 to move negations symbols inside of the formulas as possible, or to eliminate negations by 3.

Step 3. Use equivalences 6,7 to move the conjunctions outside of the disjunction.

The correctness of the algorithm can be proved by structural induction on formulae, using the fact that we always replace equivalents by equivalents. The algorithm terminates since in each step we decrease the length of the original formula.

Exercise 9

We have to prove: $\Sigma \vdash_H A$ implies $\Sigma \models A$. First we prove that every axiom in H is a tautology (give the truth tables). Now, we have to prove: if $\Sigma \vdash_H A$ implies $\Sigma \models A$. Since every proof of A in H is a sequence of formulae: $P_1, ..., P_N$, where $P_N = A$, we prove the soundness by induction on the length of the proof.

Base Case. If $N = 1$ then there are only two possibilities:

1. A is in Σ

2. A is an axiom.

In the first case, obviously: $\Sigma \models A$. In the second case, by lemma 1 we have that every model M holds $M \models A$, and therefore $\Sigma \models A$.

Inductive Hyp. Suppose that the property is true for proofs with length less than N

Inductive Case. Suppose that we have a proof of A: $P_1, P_2, ..., P_N$, where $P_N = A$, then we have three possibilities:

1. A is in Σ.

2. A is an axiom.

3. A is deduced by Modus Ponens

In the two first cases the proof is the same that the Base Case. Now, suppose that we have A by Modus Ponens, that is, there are some $P_i = (B \Rightarrow A)$ and $P_j = B$ with $i, j < N$. By Inductive Hypothesis we know:

- $\Sigma \models B$ (*)
- $\Sigma \models B \Rightarrow A$ (**)

Now, suppose that there exist some model M such as: $M \models \neg A$ and $M \models \Sigma$, but by (**) we have: $M \models B \Rightarrow A$, but then the only way is that $M \models \neg B$, but by (*) we have $M \models B$!, this is a contradiction. Then for all models $M \models \Sigma$ we have $M \models A$.

Exercise 10

a)

The easiest way is using completeness, since $\models p_0 \Rightarrow \neg\neg p_0$ we have that $\vdash p_0 \rightarrow p_0$, another way is using the deduction theorem.

b)

First we prove some lemmas:
Lemma 1 $\vdash A \Rightarrow A$

1	$[A \Rightarrow ((A \Rightarrow A) \Rightarrow A)] \Rightarrow [(A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)]$	Ax2
2	$A \Rightarrow [(A \Rightarrow A) \Rightarrow A]$	Ax1
3	$(A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)$	MP 1, 2
4	$A \Rightarrow (A \Rightarrow A)$	Ax1
5	$A \Rightarrow A$	MP 3, 4

Lemma 2: $\vdash \neg\neg A \Rightarrow A$:

1	$\neg\neg A \rightarrow (\neg\neg\neg\neg A \Rightarrow \neg\neg A)$	Ax1
2	$(\neg\neg\neg\neg A \rightarrow \neg\neg A) \Rightarrow (\neg A \Rightarrow \neg\neg\neg A)$	Ax3
3	$[(\neg\neg\neg\neg A \Rightarrow \neg\neg A) \rightarrow (\neg A \Rightarrow \neg\neg\neg A)] \Rightarrow [\neg\neg A \Rightarrow [(\neg\neg\neg\neg A \Rightarrow \neg\neg A) \rightarrow (\neg A \Rightarrow \neg\neg\neg A)]]$	Ax1
4	$[\neg\neg A \Rightarrow [(\neg\neg\neg\neg A \Rightarrow \neg\neg A) \rightarrow (\neg A \Rightarrow \neg\neg\neg A)]]$	MP 2, 3
5	$[\neg\neg A \Rightarrow [(\neg\neg\neg\neg A \rightarrow \neg\neg A) \Rightarrow (\neg A \Rightarrow \neg\neg\neg A)]] \Rightarrow [(\neg\neg A \Rightarrow (\neg\neg\neg\neg A \Rightarrow \neg\neg A)) \Rightarrow (\neg\neg A \Rightarrow (\neg A \Rightarrow \neg\neg\neg A))]$	Ax2
6	$[(\neg\neg A \Rightarrow (\neg\neg\neg\neg A \Rightarrow \neg\neg A)) \Rightarrow (\neg\neg A \Rightarrow (\neg A \Rightarrow \neg\neg\neg A))]$	MP 4,5
7	$\neg\neg A \Rightarrow (\neg A \Rightarrow \neg\neg\neg A)$	MP 1, 6
8	$(\neg A \Rightarrow \neg\neg\neg A) \Rightarrow (\neg\neg A \Rightarrow A)$	Ax3
9	$[(\neg A \Rightarrow \neg\neg\neg A) \Rightarrow (\neg\neg A \Rightarrow A)] \Rightarrow [\neg\neg A \Rightarrow [(\neg A \rightarrow \neg\neg\neg A) \Rightarrow (\neg\neg A \Rightarrow A)]]$	Ax1
10	$\neg\neg A \Rightarrow [(\neg A \Rightarrow \neg\neg\neg A) \Rightarrow (\neg\neg A \Rightarrow A)]$	MP 8, 9
11	$[\neg\neg A \Rightarrow [(\neg A \Rightarrow \neg\neg\neg A) \Rightarrow (\neg\neg A \Rightarrow A)]] \Rightarrow [[\neg\neg A \Rightarrow (\neg A \Rightarrow \neg\neg\neg A)] \Rightarrow [\neg\neg A \Rightarrow (\neg\neg A \Rightarrow A)]]$	Ax2
12	$[\neg\neg A \Rightarrow (\neg A \Rightarrow \neg\neg\neg A)] \Rightarrow [\neg\neg A \Rightarrow (\neg\neg A \Rightarrow A)]$	MP 10, 11
13	$\neg\neg A \Rightarrow (\neg\neg A \Rightarrow A)$	MP 7, 12
14	$[\neg\neg A \Rightarrow (\neg\neg A \Rightarrow A)] \Rightarrow [(\neg\neg A) \Rightarrow (\neg\neg A \Rightarrow A)]$	Ax2
15	$(\neg\neg A \Rightarrow \neg\neg A) \Rightarrow (\neg\neg A \Rightarrow A)$	MP 13, 14
16	$\neg\neg A \Rightarrow \neg\neg A$	Lemma 1
17	$\neg\neg A \Rightarrow A$	MP 15, 16

Using this we prove $A \Rightarrow \neg\neg A$ as follows:

1	$(\neg\neg\neg A \Rightarrow \neg A) \Rightarrow (A \Rightarrow \neg\neg A)$	Ax3
2	$\neg\neg\neg A \Rightarrow \neg A$	Lemma 2
3	$A \Rightarrow \neg\neg A$	MP 1,2

Exercise 11

Recall that a set of formulas Σ is consistent iff there is no a formula A such as $\Sigma \vdash_H A$ and $\Sigma \vdash_H \neg A$. We have to prove:

A set of formulas Σ is consistent if and only if Σ is satisfiable.

Proof.
if) If Σ is satisfiable then Σ is consistent. We prove this by contradiction.

Suppose that Σ is satisfiable, then must be some model M such that $M \vDash \Sigma$. If Σ is inconsistent then:

- $\Sigma \vdash_H A$, and
- $\Sigma \vdash_H \neg A$

For some A. By exercise 11 (soundness) we have:

- $M \vDash A$, and
- $M \vDash \neg A$

which is a contradiction!. Then Σ must be consistent.

Only if) We have to prove: If Σ is consistent then Σ is satisfiable.

Suppose that Σ is consistent and Σ is not satisfiable, then by definition of $\vDash$ and since Σ doesn't have any model it holds that:

- $\Sigma \vDash A$ and,
- $\Sigma \vDash \neg A$

but this implies, by completeness, that: $\Sigma \vdash_H A$ and $\Sigma \vdash_H \neg A$, this is a contradiction