# Solutions of Exercise Group Number 2
# Logic and Discrete Mathematics in Software Engineering
# Fall 2008

## Exercise 1

$\overline{A \cup B}$
$=$ { definition of complement }
$\{x \mid x \in U : \neg(x \in (A \cup B))\}$
$=$ { definition of $\notin$ }
$\{x \mid x \in U : \neg(x \in A \vee x \in B)\}$
$=$ { DeMorgan for Propositional Logic }
$\{x \mid x \in U : \neg(x \in A) \wedge \neg(x \in B)\}$
$=$ { definition of $\notin$ }
$\{x \mid x \in U : x \notin A \wedge x \notin B\}$

The other law is exactly the same idea (*mutatis mutandis*).

## Exercise 2

$(A \cap B) \cup (A \cap \overline{B})$
$=$ { distributivity $\cup$, $\cap$ }
$A \cap (B \cup \overline{B})$
$=$ { complement properties }
$A \cap U$
$=$ { $A \subseteq U$ }
$A$

# Exercise 3

$$
\begin{array}{ll}
A \oplus B & \\
= & \{ \text{ definition of } \oplus \} \\
(A \cup B) - (A \cap B) & \\
= & \{ \text{ definition of } - \} \\
(A \cup B) \cap (\overline{A \cap B}) & \\
= & \{ \text{ DeMorgan} \} \\
(A \cup B) \cap (\overline{A} \cup \overline{B}) & \\
= & \{ \text{ Distributivity } \} \\
(A \cap (\overline{A} \cup \overline{B}) \cup (B \cap (\overline{A} \cup \overline{B})) & \\
= & \{ \text{ Distributivity } \} \\
((A \cap \overline{A}) \cup (A \cap \overline{B})) \cup ((B \cap \overline{A}) \cup (B \cap \overline{B})) & \\
= & \{ \text{ properties of complement and union } \} \\
(A \cap \overline{B}) \cup (B \cap \overline{A}) & \\
= & \{ \text{ definition of } - \} \\
(A - B) \cup (B - A) &
\end{array}
$$

# 1 Exercise 4

## a)

$f(x) = -3x + 4$ is a **bijection**.
**Proof.**
Injectivity)
We have to prove: $f(x) = f(y) \Rightarrow x = y$ (contrapositive of $x \neq y \Rightarrow f(x) \neq f(y)$).

$$
\begin{array}{ll}
f(x) = f(y) & \\
\Rightarrow & \{\text{definition of f}\} \\
-3x + 4 = -3y + 4 & \\
\Rightarrow & \{\text{Arithmetic}\} \\
x = y &
\end{array}
$$

Onto)
We have to prove: $\forall x : \exists y : f(y) = x$. Let $x \in \mathbb{R}$ be then:

$$x = (-3 * \frac{x-4}{-3}) + 4 = f(\frac{x-4}{-3})$$

□

## b)

$f(x) = -3x^2 + 7$. This function is not bijective because it is not surjective. For example, there is no $x$ such that: $f(x) = 10$. Otherwise:

$$
\begin{aligned}
& f(x) = 10 \\
\Rightarrow \quad & \{\text{definition of } f\} \\
& -3x^2 + 7 = 10 \\
\Rightarrow \quad & \{\text{Arithmetic}\} \\
& -3x^2 = 3 \\
\Rightarrow \quad & \{\text{Arithmetic}\} \\
& x^2 = -1!
\end{aligned}
$$

which is a contradiction (we need complex numbers for such a solution).

## c)

$$f(x) = \frac{x+1}{x+2}$$

This function is not bijective, because it is not a surjective function. For example does not exist some $x$ such that $f(x) = 1$. Otherwise:

$$1 = \frac{x+1}{x+2} \Leftrightarrow x + 2 = x + 1 \Leftrightarrow 2 = x + 1 - x \Leftrightarrow 2 = 1$$

which is a contradiction.

## d)

$f(x) = x^5 + 1$ is a bijective function.
**Proof.**
Injectivity)

$$
\begin{aligned}
& f(x) = f(y) \\
\Rightarrow \quad & \{\text{definition of } f\} \\
& x^5 + 1 = y^5 + 1 \\
\Rightarrow \quad & \{\text{Arithmetic}\} \\
& x^5 = y^5 \\
\Rightarrow \quad & \{\text{Arithmetic}\} \\
& \sqrt[5]{x^5} = \sqrt[5]{y^5} \\
\Rightarrow \quad & \{\text{Arithmetic, } \sqrt[5]{\ } \text{ preserves signs}\} \\
& x = y
\end{aligned}
$$

Surjectivity) Let $x$ be an element of $\mathbb{R}$ then:

$$x = (\sqrt[5]{x-1})^5 + 1 = f(\sqrt[5]{x-1})$$

# Exercise 5

## a)

We have to prove: $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$. Recall the definition of inverse image is:

$$f^{-1}(S) = \{a \in A \mid f(a) \in S\}$$

We prove this showing that: $a \in f^{-1}(S \cup T) \Leftrightarrow a \in f^{-1}(S) \cup f^{-1}(T)$,
**Proof.**

$$\begin{array}{ll}
a \in f^{-1}(S \cup T) & \\
\Leftrightarrow & \{\text{definition of } f^{-1}\} \\
f(a) \in S \cup T & \\
\Leftrightarrow & \{\text{definition of } \cup\ \} \\
f(a) \in S \vee f(a) \in T & \\
\Leftrightarrow & \{\text{definition of } f^{-1}\} \\
a \in f^{-1}(S) \vee a \in f^{-1}(T) & \\
\Leftrightarrow & \{\text{definition of } \cup\} \\
a \in f^{-1}(S) \cup f^{-1}(T) &
\end{array}$$

## b)

We have to prove: $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$, we prove this showing that:

$$a \in f^{-1}(S \cap T) \Leftrightarrow a \in f^{-1}(S) \cap f^{-1}(T)$$

**Proof.**

$$\begin{array}{ll}
a \in f^{-1}(S \cap T) & \\
\Leftrightarrow & \{\text{definition of } f^{-1}\} \\
f(a) \in S \cap T & \\
\Leftrightarrow & \{\text{definition of } \cap\} \\
f(a) \in S \wedge f(a) \in T & \\
\Leftrightarrow & \{\text{definition of } f^{-1}\} \\
a \in f^{-1}(S) \wedge a \in f^{-1}(T) & \\
\Leftrightarrow & \{\text{definition of } \cap\} \\
a \in f^{1}(S) \cap f^{-1}(T) &
\end{array}$$

# Exercise 6

Consider the following matrices:

$$M_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

then:

## a)

$$M_{R_1 \cup R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

## b)

$$M_{R_1 \cap R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

## c)

$$M_{R_2 \circ R_1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

## d)

$$M_{R_1 \circ R_1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

## e)

$$M_{R_1 \oplus R_2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

# Exercise 7

Let $R$ be the following relation: $(a, b)R(c, d) \Leftrightarrow a * d = b * c$. We have to prove that $R$ is a equivalence relation.
**Proof.**

**Reflexivity)** By arithmetic we know $a * b = a * b$ but this implies $(a, b)R(a, b)$

**Symmetry)**

$$
\begin{array}{ll}
(a,b)R(c,d) & \\
\Rightarrow & \{\text{definition of } R\} \\
a*d = b*c & \\
\Rightarrow & \{\text{symmetry of } = \text{ and commutativity of } *\} \\
c*b = d*a & \\
\Rightarrow & \{\text{definition of } R\} \\
(c,d)R(a,b) &
\end{array}
$$

**transitivity)** Suppose $(a,b)R(c,d) \wedge (c,d)R(e,f)$ by definition of $R$ it holds that:

1. $a*d = b*c$
2. $c*f = d*e$

If we multiply $a$ in both sides of (2) we obtain:

$$a*c*f = a*d*e$$

But by (1) we can replace $a*d$ by $b*c$ then we obtain:

$$a*c*f = b*c*e$$

Then we can eliminate $c$ (recall that $c \neq 0$) from both sides, and then:

$$a*f = b*e$$

But by definition of $R$ this means: $(a,b)R(e,f)$.

# Exercise 8

## a)

The class of equivalence is $[(1,2)]_R = \{(a,b) \in R \mid b = 2a\}$

## b)

The class of equivalence of $(a,b)$ are the numbers $\dfrac{c}{d}$ (represented as $(c,d)$) such that $\dfrac{a}{b} = \dfrac{c}{d}$

# Exercise 9

## a)

Integers divisible by 3. This set is countable. Let us call $\mathbb{Z}_{-3}$ the set of integers not divisible by 3, note that the number 0 does not belong to this set because: $0 * 3 = 0$. The following one-to-one function maps each natural to one element of $\mathbb{Z}^+_{-3}$ (the positive part of $\mathbb{Z}_{-3}$).

$$f(x) = \lfloor \frac{3}{2} * x \rfloor + 1$$

This function works as follows:

| $\mathbb{N}$ | $f$ | $\mathbb{Z}^+_{-3}$ |
|---|---|---|
| 0 | $\leftrightarrow$ | 1 |
| 1 | $\leftrightarrow$ | 2 |
| 2 | $\leftrightarrow$ | 4 |
| 3 | $\leftrightarrow$ | 5 |
| 4 | $\leftrightarrow$ | 7 |
| | . | |
| | . | |
| | . | |

Defining a bijective function $f' : \mathbb{Z}^+_{-3} \rightarrow \mathbb{Z}_{-3}$ is straightforward, and therefore we obtain a new bijection $f' \circ f : \mathbb{N} \rightarrow \mathbb{Z}_{-3}$.

## b)

The following function maps the natural to the positive part of the required set.

$$f(x) = 5 * (\lfloor \frac{7}{6} * x \rfloor + 1)$$

This function works as follows:

| $\mathbb{N}$ | $g$ | $\mathbb{F}$ |
|---|---|---|
| 0 | $\leftrightarrow$ | 5 |
| 1 | $\leftrightarrow$ | 10 |
| 2 | $\leftrightarrow$ | 15 |
| 3 | $\leftrightarrow$ | 20 |
| 4 | $\leftrightarrow$ | 25 |
| 5 | $\leftrightarrow$ | 30 |
| 6 | $\leftrightarrow$ | 40 |
| | . | |
| | . | |
| | . | |

As we did above, it is straightforward to extend this function for the negative part of this set.

## c)

Real numbers with decimal representation with all 1's. This set is **countable**. We call this set $U$. First note that we can map each (finite) real number with decimal representation of 1's, to tuples $(n, m)$ where $n$ is the number of 1's in $x$ at the right of the point, and $m$ is the number at the left. The problem is when the right part is infinite, we fix this adding one to the second component, i.e., we define the following function: $f : U \to \mathbb{N} \times \mathbb{N}$, as follows:

$$f(x) = \begin{cases} (n, 0) & \text{if the left part of } x \text{ has } n \text{ 1's and the right part is infinite} \\ (n, m+1) & \text{if the left part of } x \text{ has } n \text{ 1's and the right has } m \text{ 1's} \end{cases}$$

For example:

| $U$ | $f$ | $\mathbb{N} \times \mathbb{N}$ |
|---|---|---|
| 1 | $\leftrightarrow$ | $(1, 1)$ |
| 11 | $\leftrightarrow$ | $(2, 1)$ |
| 1.111... | $\leftrightarrow$ | $(1, 0)$ |
| 11.1111... | $\leftrightarrow$ | $(2, 0)$ |
| 11.11 | $\leftrightarrow$ | $(2, 3)$ |
| | . | |
| | . | |
| | . | |

then we have to show that $\mathbb{N} \times \mathbb{N}$ is numerable, we can do that using the following matrix:

| | 0 | 1 | 2 | 3 | ... |
|---|---|---|---|---|---|
| 0 | $(0,0)$ | $(0,1)$ | $(0,2)$ | $(0,3)$ | ... |
| 1 | $(1,0)$ | $(1,1)$ | $(1,2)$ | $(1,3)$ | ... |
| 2 | $(2,0)$ | $(2,1)$ | $(2,2)$ | $(2,3)$ | ... |
| . | | | | | ... |
| . | | | | | ... |
| . | | | | | |

And then we can use the Cantor numeration with this matrix, that is:

$$(0, 0), (0, 1), (1, 0), (2, 0), (1, 1), (0, 2), ...$$

□

## d)

Real number with decimal representation with all 1's or 9's. This set is not countable. To prove this we can use the diagonal argument, proving that a subset of this set is no countable. Consider:

$$R = \{1.d_1d_2d_3.... \mid d_i \in \{1, 9\}\}$$

Suppose that we can enumerate this set, then we can list its elements in the following way:

$$
\begin{array}{l}
1.d_{1,1}\ d_{1,2}\ d_{1,3}\ d_{1,4}.....\\
1.d_{2,1}\ d_{2,2}\ d_{2,3}\ d_{2,4}.....\\
1.d_{3,1}\ d_{3,2}\ d_{3,3}\ d_{3,4}.....\\
.\\
.\\
.\\
.\\
.
\end{array}
$$

Now, we can consider the number:

$$e = 1.\overline{d_{1,1}}\ \overline{d_{2,2}}\ \overline{d_{3,3}}\ \overline{d_{4,4}}....$$

where:

$$\overline{d_{i,i}} = \begin{cases} 1 & \text{if } d_{i,i} = 9 \\ 9 & \text{if } d_{i,i} = 1 \end{cases}$$

This number is different that each number listed before because it differs by one digit with each of them! I.e., we have found a number in $R$ not listed, this is a contradiction, then $R$ is not enumerable.

# Exercise 10

Consider the solutions of equations of type: $ax^2 + bx + c = 0$, the set of real numbers which are solutions of these equations is countable. First, every solution $x$ of this equation could be calculate with the following formula:

$$x = \frac{b\Box\sqrt[2]{b^2 - 2 * a * c}}{2 * a}$$

Where $\Box = -/+$. This implies that we can enumerate these $x$'s using tuples $(a, b, c)$. The if we prove that $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ is numerable then is straightforward that the set of solution is numerable. But we know that $\mathbb{Z}$ is equipollent to $\mathbb{N}$ then we only have to prove that $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ is numerable. To see this, first we prove that $\mathbb{N}\times\mathbb{N}$ is numerable, the proof is the standard one, consider the following matrix:

| | 0 | 1 | 2 | 3 | ... |
|---|---|---|---|---|---|
| 0 | $(0,0)$ | $(0,1)$ | $(0,2)$ | $(0,3)$ | ... |
| 1 | $(1,0)$ | $(1,1)$ | $(1,2)$ | $(1,3)$ | ... |
| 2 | $(2,0)$ | $(2,1)$ | $(2,2)$ | $(2,3)$ | ... |
| . | | | | | ... |
| . | | | | | ... |
| . | | | | | |

Then the enumeration is: $(0,0), (0,1), (1,0), (2,0), (1,1), (0.2)....$ We call this function $f$. Now using $f$ we can build a function one-to-one from $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ to $\mathbb{N}\times\mathbb{N}$, given the tuple $(a, b, c)$ we only maps it to $(a, f(b, c))$, since $f$ is one-to-one the new function is one-to-one too.

# Exercise 11

Given a relation $R \subseteq A \times B$, we define the function $f_R$ as follows:

$$f_R(x) = \{y \in B \mid xRy\}$$

# Exercise 12

Consider the set $F = \{f \mid f : \mathbb{N} \rightarrow \mathbb{N}\}$, this set is not countable, it has the cardinality of real numbers. To see this, we can show a one-to-one function between functions in $F$ and the numbers in the interval $[0, 1)$, by Cantor's theorem we know that this interval has the same cardinality that real numbers. We define the following one-to-one function $\chi : F \rightarrow [0, 1)$, as follows:

$$\chi(f) = 0.f(0)f(1)(2)f(3)f(4)...$$

We can proof that this function is a bijection:
**Proof.**
Injectivity) Suppose two functions $f \neq g$, that means that exists some $x$ such that: $f(x) \neq g(x)$, but this means that $\chi(f) \neq \chi(g)$, since them differ in their $xth$ element.

Surjectivity) Suppose a number $x = 0.d_0d_1d_2...$ then we can build the function: $f(i) = d_i$.

# Exercise 13

Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be total, and let $g \circ f : A \rightarrow C$.

## a)

We have to prove that if $f$ and $g$ are injective than also $g \circ f$ is injective.

**Proof.** We have to prove: $x \neq y \Rightarrow g \circ f(x) \neq g \circ f(y)$.

$$\begin{array}{ll} x \neq y & \\ \Rightarrow & \{f \text{ is injective}\} \\ f(x) \neq f(y) & \\ \Rightarrow & \{\text{g is injective}\} \\ g(f(x)) \neq g(f(y)) & \\ \Rightarrow & \{\text{definition of } \circ\} \\ g \circ f(x) \neq g \circ f(y) & \end{array}$$

□

The **converse** of this property is not true. We can show this by means of a counterexample, consider the following scenario:

- $A = \{a\}$

- $B = \{b, c\}$
- $C = \{d\}$
- $f(a) = b$
- $g(b) = d$ and $g(c) = d$

then $g \circ f$ is injective but $g$ is not injective.

### b)

We have to prove: $\forall c \in C : \exists a \in A : g \circ f(a) = c$. We know that $f$ and $g$ are injective. Let $c \in C$ be, then by hypothesis:

$$\exists b \in B : g(b) = c \tag{1}$$

but also:

$$\exists a \in A : f(a) = b \tag{2}$$

But then using 1 and 2 and replacing $f(a)$ by $b$ we obtain:

$$\exists a \in A : g(f(a)) = c$$

□

The converse is false in general. The counterexample given in item a) works, in that scenario $g \circ f$ is surjective but $f$ is not surjective.

## Exercise 14

### a)

We can enumerate the nodes using binary numbers, 1 to the root, and then, let $n$ be the binary number of some node, the number of the left child is $n0$ (the number $n$ followed by 0), and the number of the right child is $n1$. The total of binary numbers that we obtain is $2^{h+1} - 1$, where $h$ is the height. The cardinality of the set of paths is the same that the set of leaves which is $2^h$.

### 1.1 b)

Using the same enumeration of exercise b, we can map each node to a natural number, and therefore the cardinality of this set is $\aleph_0$. On the other hand, the set of paths is of cardinality $\aleph_1$, this can be proved using the fact that each path is an infinite sequence of binary numbers, and then we can give a bijection between this set and the interval $[0, 1)$, which is of cardinality $\aleph_1$.