Solutions of Exercise Group Number 3
Logic and Discrete Mathematics in Software Engineering
Fall 2008

Exercise 1

(a) The maximal elements are l and m.

(b) The minimal elements are a, b and c.

(c) There is no greatest element.

(d) There is no least element.

(e) The upper bounds of $\{a, b, c\}$ are k, l and m.

(f) There is no least bounds of $\{f, g, h\}$.

(g) There is no greatest lower bound for $\{f, g, h\}$.

Exercise 2

(a) The minimal elements are 2 and 9.

(b) There is no greatest element.

(c) There is no least element.

(d) The upper bounds of $\{2, 9\}$ are $\{18, 36, 72\}$.

(e) The least upper bound of $\{2, 9\}$ is 18.

(f) The lower bounds of $\{60, 72\}$ are $\{2, 4, 6, 12\}$.

(g) The greatest lower bound for $\{60, 72\}$ is 12.

Exercise 3

(a) $(\exists x : P(x, 3))$ can be written as $(P(1, 3) \vee P(2, 3) \vee P(3, 3))$

(b) $(\forall x : P(1, y))$ can be written as $(P(1, 1) \wedge P(1, 2) \wedge P(1, 3))$.

(c) $(\exists y : \neg P(2, y))$ can be written as $(\neg P(2, 1) \vee \neg P(2, 2) \vee \neg P(2, 3))$.

(d) $(\forall x : \neg P(x, 2))$ can be written as $\neg P(1, 2) \wedge \neg P(2, 2) \wedge \neg P(3, 2)$.

Exercise 4

We have to show that $(\forall x : P(x)) \vee (\forall x : Q(x))$ and $\forall x : (P(x) \vee Q(x))$ are not logical equivalent. With this goal in mind we show a model which satisfies the second but not the first formula. Let $M = (D, I)$ be the following model:

- $D = \{1, 2\}$, and
- $I(P) = \{(1, T), (2, F)\}$
- $I(Q) = \{(1, F), (2, T)\}$

Then:

$$M \vDash \forall x : (P(x) \vee Q(x))$$

but:

$$M \nvDash (\forall x : P(x)) \vee (\forall x : Q(x))$$

Exercise 5

Consider the following predicates:

- $I(x)$: *x has a internet connection.*
- $C(x, y)$: *x and y have chatted over the internet.*

a)

Jerry does not have an internet connection. Formalization:

$$\neg I(Jerry)$$

b)

Rachel has not chatted over the internet with Chelsea. Formalization:

$$\neg C(Rachel, Chelsea)$$

c)

Jan and Sharon have never chatted. Formalization:

$$\neg C(Jan, Sharon)$$

d)

No one in the class has chatted with Bob. Formalization:

$$\forall x : \neg C(x, Bob)$$

e)

Sanjay has chatted with everyone except Joseph. Formalization:

$$(\forall x : x \neq Joseph \Rightarrow C(Sanjay, x)) \wedge \neg C(Sanjay, Joseph)$$

f)

Someone in your class does not have internet connection. Formalization:

$$\exists x : \neg I(x)$$

g)

Not everyone in your class has an internet connection. Formalization:

$$\neg \forall x : I(x)$$

h)

Exactly one student in your class has an internet connection. Formalization:

$$\exists x : (I(x) \wedge (\forall y : y \neq x \Rightarrow \neg I(x)))$$

i)

Everyone except one student in your class has internet connection. Formalization:

$$\exists x : (\neg I(x) \wedge (\forall y : y \neq x \Rightarrow I(x)))$$

j)

Everyone in your class with internet connection has chatted over the internet with at least one other student in the class. Formalization:

$$\forall x : I(x) \Rightarrow (\exists y : x \neq y \wedge C(x, y))$$

k)

Someone in your class has an internet connection but has not chatted with anyone else in your class. Formalization:

$$\exists x : (I(x) \wedge (\forall y : \neg C(x, y)))$$

l)

There are two student in the class who have not chatted with each other over the internet. Formalization:

$$\exists x : \exists y : x \neq y \wedge \neg C(x, y)$$

m)

There is a student in your class who has chatted with everyone in your class over the internet. Formalization:

$$\exists x : \forall y : C(x, y)$$

n)

There are at least two student in your class who have not chatted with the same person in the class. Formalization:

$$\exists x : \exists y : x \neq y \wedge \neg(\exists z : C(x, z) \wedge C(y, z))$$

o)

There are two students in the class who between them have chatted with everyone else in the class. Formalization:

$$\exists x : \exists y : (x \neq y) \wedge (\forall z : (C(x, z) \vee C(y, z)))$$

Exercise 6

We add two predicates to distinguish between natural numbers and real numbers. Let $R(x)$ be an unary predicated which, intuitively, says x *is a real number*, and let $N(x)$ be an unary predicated which, intuitively says x *is a natural number*, then the statement $\lim_{n\to\infty} a_n = L$, can be expressed in FOL as:

$$R(L) \wedge (\forall \epsilon \cdot R(\epsilon) \Rightarrow (\exists n_0 \cdot N(n_0) \wedge (\forall n \cdot N(n) \wedge R(a_n) \wedge n > n_0 \Rightarrow |a_n - L| < \epsilon)))$$

where L is a constant, and a_n denotes the term $a(n)$ being a a function symbol of arity 1.

Exercise 7

Given $\langle S_1, \leq_1\rangle$ and $\langle S_2, \leq_2\rangle$ partial orders, we have to show that $\langle S_1 \times S_2, \leq\rangle$ is a partial order, where:

$$\langle a, x\rangle \leq \langle b, y\rangle \Leftrightarrow a \leq_1 b \wedge x \leq_2 y$$

Reflexivity. Since $\leq_1$ and $\leq_2$ are partial orders we know that $a \leq_1 a$ and $x \leq_2 x$, and therefore by definition of $\leq$ we obtain $\langle a, x\rangle \leq \langle a, x\rangle$.

Antisymmetry. Suppose that $\langle a, x\rangle \leq \langle b, y\rangle$ and $\langle b, y\rangle \leq \langle a, x\rangle$, by definition of $\leq$ we obtain $a \leq_1 b$, $b \leq_1 a$, $x \leq_2 y$ and $y \leq_2 x$, using these inequations and the fact that $\leq_1$ and $\leq_2$ are partial orders we obtain $a = b$ and $x = y$, and therefore by definition of pairs we have that: $\langle a, x\rangle = \langle b, y\rangle$.

Transitivity. Suppose that $\langle a, x\rangle \leq \langle b, y\rangle$ and $\langle b, y\rangle \leq \langle c, w\rangle$. Then using the definition of $\leq$ we obtain $a \leq_1 b \leq_1 c$ and $x \leq_2 y \leq_2 w$, since $\leq_1$ and $\leq_2$ are partial orders we have that $a \leq_1 c$ and $x \leq_2 w$, from here using the definition of $\leq$ we get $\langle a, x\rangle \leq \langle c, w\rangle$.

Exercise 8

We define:

- $a \wedge b = \mathsf{min}(a, b)$
- $a \vee b = \mathsf{max}(a, b)$

Since $\langle L, \leq\rangle$ is a total order we know that the minimum and the maximum between two elements there exists. Now, suppose that $a \leq c$ and $b \leq c$, since the maximum between a and b is one of them we get $\mathsf{max}(a, b) \leq c$. On the other hand if $c \leq a$ and $c \leq b$ we have that $c \leq \mathsf{min}(a, b)$ since the minimum between a and b is also one of them. Summarizing, $a \wedge b$ is the greater lower bound between a and b, and $a \vee b$ is the least upper bound.

Exercise 9

A formula which has the form:

$$Q_1 x_1 : ... : Q_N : x_N : \phi$$

where each $Q_i x_i$ is either an universal or an existential quantifier (where $x_1 \neq x_j$ for $i \neq j$), and the formula ϕ is a quantifier free formula, is in **Prenex Normal Form**.

Consider the following equivalences:

1. $\phi \Leftrightarrow \psi \equiv (\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi)$
2. $\phi \Rightarrow \psi \equiv \neg\phi \vee \psi$
3. $\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$
4. $\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$

5. $\neg(\forall x : \phi) \equiv \exists x : \neg\phi$

6. $\neg(\exists x : \phi) \equiv \forall x : \neg\phi$

7. $\psi \wedge \forall x : \phi \equiv \forall x : (\psi \wedge \phi)$, if x does not occur in ϕ.

8. $\psi \vee \forall x : \phi \equiv \forall x : (\psi \vee \phi)$, if x does not occur in ϕ.

9. $\psi \wedge \exists x : \phi \equiv \exists x : (\psi \wedge \phi)$, if x does not occur in ϕ.

10. $\psi \vee \exists x : \phi \equiv \exists x : (\psi \vee \phi)$, if x does not occur in ϕ.

then the algorithm to translate a formula to *Prenex Normal Form* is:

step 1 Remove all $\Rightarrow$ and $\Leftrightarrow$ using rules 1,2.

step 2 Push all $\neg$ inside of the formula as possible using rules: 2-5

step 3 Rename bound and free variables in such way that no two quantifiers bind the same variable, and no variable has both free and bound occurrences.

step 4 Use rules 6-9 to pull out the quantifiers. (By step 3 we can do it)

We can prove that given a formula ϕ this algorithm returns a formula ϕ' such that $\phi \Leftrightarrow \phi'$. The proof is by induction on the formula. We call $A(\phi)$ to the application of the algorithm to formula ϕ.
Base Case. The base case is when ϕ is $p(t_1, ..., t_n)$ being p a predicate symbol. Actually, this formula is in prenex normal form, and the algorithm returns $\phi' = \phi$, then $\phi \Leftrightarrow \phi'$.
Inductive case) We have the following cases:
If $\phi = \neg\psi$ where:

- $\psi = \neg\psi'$, in this case using the step 1 the algorithm returns $A(\psi')$ which is in Prenex Normal Form, and equivalent to the original formula, by inductive hypothesis.

- $\psi = \psi' \Rightarrow \psi''$ in this case the algorithm perform steps 1 and 2 obtaining: $\neg\psi' \vee \psi''$, but since $A(\neg\psi')$ and (ψ'') are in Prenex Normal form by inductive hypothesis, using steps 3 and 4, the algorithm returns a formula in Prenex Normal Form.

- $\psi = \forall x : \psi'$, using the algorithm we obtain a formula in prenex normal form $A(\neg\psi')$, but then the algorithm returns $\exists x : A(\neg\psi')$ which is in PNF.

If $\phi = \psi \rightarrow \psi'$, then we have $A(\neg\psi)$ and $A(\psi')$ in PNF and then using steps 3,4 on $A(\neg\psi) \vee A(\psi)$ we obtain a PNF formula.

If $\phi = \forall x : \psi$, we obtain a PNF formula $A(\psi)$ and then $\forall x : A(\psi)$ is in PNF.

□

Exercise 10

The alphabet of L is $\langle \emptyset, \emptyset, \leq \rangle$, and we have the following set of axioms:
$\{\forall x \cdot x \leq x,$
$\forall x, y \cdot x \leq y \wedge y \leq x \Rightarrow x = y,$
$\forall x, y, z \cdot x \leq y \wedge y \leq z \Rightarrow x \leq z,$
$\forall x, y \cdot (\exists l \cdot (l \leq x \wedge l \leq y \wedge (\forall z \cdot z \leq x \wedge z \leq y \Rightarrow z \leq j))),$
$\forall x, y \cdot (\exists u \cdot (x \leq u \wedge y \leq u \wedge (\forall z \cdot x \leq z \wedge y \leq z \Rightarrow u \leq z)))\}$

Exercise 11

The trick in this exercise is to add new predicate symbols to distinguish between vectors and scalars. The language of the theory of vector space over fields is defined as follows:

- $C = \{0_s, 1_s, 0_v\}$
- $F = \{+_v, +_s, \times_s, \times\}$, where the arity of these functions is 2.
- $P = \{V, S\}$ the arity of V and S is 1.

The following is the set of axioms:
$\{V(0_v),$
$S(1_s), S(0_s)$
$\forall x, y \cdot S(x) \wedge S(y) \Rightarrow S(x \times_s y) \wedge S(x +_s y),$
$\forall a, b \cdot V(a) \wedge V(b) \Rightarrow V(a +_v b),$
$\forall x, a \cdot S(x) \wedge V(a) \Rightarrow V(x \times a),$
$\forall x, y, z \cdot S(x) \wedge S(y) \wedge S(z) \Rightarrow x +_s (y +_s z) = (x +_s y) +_s z,$
$\forall x, y \cdot S(x) \wedge S(y) \Rightarrow x +_s y = y +_s x,$
$\forall x, y, z \cdot S(x) \wedge S(y) \wedge S(z) \Rightarrow x \times_s (y +_s z) = (x \times_s y) +_s (x \times_s z),$
$\forall x \cdot S(x) \Rightarrow x +_s 0_s = x,$
$\forall x \cdot S(x) \Rightarrow \exists x' \cdot S(x') \wedge x +_s x' = 0,$
$\forall x, y, z \cdot S(x) \wedge S(y) \wedge S(z) \Rightarrow x \times_s (y \times_s z) = (x \times_s y) \times_s z,$
$\forall x, y \cdot S(x) \wedge S(y) \Rightarrow x \times_s y = ytimes_s x,$
$\forall x, y, z \cdot S(x) \wedge S(y) \wedge S(z) \Rightarrow x +_s (y \times_s z) = (x +_s y) \times_s (x +_s z),$
$\forall x \cdot S(x) \rightarrow x \times_s 1_s = x,$
$\forall x \cdot S(x) \wedge x \neq 0 \Rightarrow \exists x' \cdot S(x') \wedge x \times_s x' = 1_s,$
$\forall a, b \cdot V(a) \wedge V(b) \Rightarrow a +_v b = b +_v a,$
$\forall a, b, c \cdot V(a) \wedge V(b) \wedge V(c) \Rightarrow a +_v (b +_v c) = (a +_v b) +_v c,$
$\forall a \cdot V(a) \Rightarrow a +_v 0_v = a,$
$\forall a \cdot V(a) \Rightarrow \exists a' \cdot V(a') \wedge a +_v a' = 0_v,$
$\forall a, x, y \cdot V(a) \wedge S(x) \wedge S(y) \Rightarrow a \times (x +_s y) = (a \times x) +_v (z \times y),$
$\forall a, b, x \cdot V(a) \wedge V(b) \wedge S(x) \Rightarrow x \times (a +_v b) = (x \times a) +_v (x \times b),$
$\forall x, y, a \cdot S(x) \wedge S(y) \wedge S(a) \Rightarrow x \times (y \times a) = (x \times_s y) \times a,$
$\forall a \cdot V(a) \Rightarrow a \times 1_s = a$
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Exercise 12

First we can define the following formulae:

- $\phi_1 \equiv (\exists x_1, x_2 : x_1 \neq x_2)$
- $\phi_2 \equiv (\exists x_1, x_2, x_3 : x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3)$
- $\phi_3 \equiv (\exists x_1, x_2, x_3, x_4 : x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_3 \neq x_4 \wedge x_1 \neq x_3 \wedge x_1 \neq x_4 \wedge x_2 \neq x_4)$
- ...
- $\phi_n \equiv (\exists x_1, ..., x_{n+1} : x_1 \neq x_2 ... \wedge x_i \neq x_j ... \wedge x_n \neq x_{n+1})$ where $x_i \neq x_j$

I.e., ϕ_n says *there exists n+1 different elements.*

We can consider the formula: $\phi^* = \bigcup_{i \in \mathbb{N}} \phi_i$. Now, given a theory $T = (L, \Gamma)$ with arbitrarily large finite models, we can consider the new theory: $T' = (L, \Gamma \cup \phi^*)$. Every finite subset of axioms $A \subseteq \Gamma \cup \phi^*$ has a model. This is easy to see, for an arbitrary A take the ϕ_n such that n is the maximum for the formulas $\phi \in A$, but by hypothesis there exists a model M of cardinality $n+1$ such as $M \vDash \Gamma$, and also $M \vDash A$ (by the cardinality of M). By the compactness theorem, and since every finite subset of $\Gamma \cup \phi^*$ has a model, there exists a model M' such that $M' \vDash \Gamma \cup \phi^*$,but this model cannot be finite, otherwise there exists some n such that not $M' \vDash \phi_n$! I.e. M' is infinite.

On the other hand, there is no theory with all finite models, otherwise we can use the same argument explained above to show that there exists an infinite model of this theory.

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