Ordinals

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Wed 18 Nov 2008

Ordinal Numbers

- First, second, third, fourth, ...

Ordinal Numbers

- Georg Cantor was the earliest to extend the counting numbers into infinity (Conway and Guy, 1996)
- Cantor's extension by introducing ordinal numbers:
  $0, 1, 2, \ldots$, then $\omega, \omega+1, \omega+2, \ldots$, then $\omega+\omega, \omega+\omega+1, \ldots$

- Russell's definition of ordinal numbers in Principia Mathematica (Volume III, page 18):
  "The name ordinal numbers is commonly confined to the relation-numbers of well-ordered series [...] the relation-numbers of series in general are commonly called order types". (Whitehead & Russell, 1925)

Well Order

- Well ordered sets: (Enderton, 1977)
  Structure (A,<) is a well order if it is linear order with the property that every non-empty subset of **A** has a least element.

- Well ordering theorem (Keeping, 1959)
  - Due to Zermelo.
  - "Every set can be well ordered".

Well Order (Examples)

- $\mathbb{N}$ is in its natural order **<** is well order
  0 < 1 < 2 < ...

- $\mathbb{Z}$ is not
  ... < -1 < 0 < 1 < ...
- but if $\mathbb{Z}$ is reordered **(|x| ≤ |y|)**, is a well order
  0, 1, -1, 2, -2, 3, -3 ...
- $\mathbb{Z}$ in a different order is a well order
  0 1 2 3 ... ; -1 -2 -3 ...

Well Order (Meaning)

- Meaning of well order:
  There is no infinitely decreasing sequence of elements
  $\ldots < a_{(n-2)} < a_{(n-1)} < a_{(n)}$

- More rigorously (Enderton, 1977):
  There exists no **f:** $\mathbb{N} \rightarrow A$ such that $f(n^{+}) < f(n)$

Order Isomorphism

- Monotone (order-preserving) function (Stoll, 1963):
  Let $f:X \rightarrow Y$, where $(X, <_1)$ and $(Y, <_2)$ be ordered sets. f is monotone iff: $x <_1 y$ implies $f(x) <_2 f(y)$

- Order isomorphism (Stoll, 1963):
  $(X, <_1)$ and $(Y, <_2)$ are called order isomorphic (ordinally similar) iff there is a bijective monotone function **f: X → Y**

- Uniqueness theorem (Stoll, 1963):
  If well-orders **A** and **B** are ordinally similar, then there exists a unique isomorphism between them.

Order Isomorphism (Examples)

- **A: {0, 1, 2, ...}** and **B: {0, 2, 4, 6, ...}** are order isomorphic.
  - isomorphism **f(x) = 2x**

- **A: {0, 1, 2, ...}** and **B: {..., 2, 1, 0}** are not order isomorphic.
  - Proof: If for some **a** $\in A$, **f(a) = 0**, then **f(a)** $< f(a+1)$, i.e. **0** $< f(a+1)$. This is a contradiction!

- **A: {0, 1, 2, ...}** and **B: {0, 1, 2, ... , 0, 1, 2, ...}** are not order isomorphic.
  - Proof: Again some **a** $\in A$ maps to **0** $\in B$. Then there are infinitely many elements in **A** before **a**. Contradiction!

Order Type

- Term coined by Cantor (Quine, 1963)
- Russell called it relation-number (Russell, 2007 reprint)
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- Order isomorphism is an equivalence relation on any collection of well ordered sets.
  - Reflexive: **A** is order isomorphic to itself
  - Symmetric: If f is monotone bijection, then so $\mathbf{f}^{-1}$
  - Transitive: $\mathbf{f} \circ g$ is bijection and monotone

- An equivalence class under order isomorphism is called an order type.

Ordinal Numbers Construction

- Numbers: **0, 1, 2, 3, ...**
- What is 2?
- **2** is set of two elements: **0, 1**
  2 = {0, 1}, 1 = {0}, 0 = {}
- So:
  0 = {}
  1 = {0}
  2 = {0, 1}
  ...
  $\omega = \{0, 1, 2, \ldots\} = \mathbb{N}$
- In general:
  n = {0, 1, ... , n-1}
  $\mathbf{n}^{+} = \{0, 1, \ldots, n-1, n\} = n \cup \{n\}$

Ordinal Numbers Construction

- von-Neumann construction by recursive definition (Halmos, 1960):
  Zero ordinal: **0 = {}**
  Successor ordinal: $\mathbf{n}^{+} = n \cup \{n\}$
  Limit ordinal: **α = sup {β: β < α} =** $\cup \{\beta : \beta < \alpha\}$
- Limit ordinal: has no immediate predecessor, i.e., there is no ordinal number **β** such that $\boldsymbol{\beta}^{+} = \alpha$
- Ordinal numbers are well ordered set:
  $\mathbf{0} \in 1 \in 2 \in 3 \in \ldots$
- Ordinals are well ordered by inclusion such that for any two ordinals **α, β**: **α = β or α** $\in \beta$ or $\beta \in \alpha$

Ordinal Numbers Constr. (Examples)

$0 = \{\}$
$1 = \{0\}$
$2 = \{0, 1\}$
$3 = \{0, 1, 2\}$
...
$\omega = \{0, 1, 2, ...\} = \mathbb{N}$

- ω: set of all finite ordinals
- ω: smallest infinite ordinal
- ω: first transfinite ordinal
- ω: first limit ordinal

Ordinal Numbers Describe Order Types

- Russell's definition of ordinal numbers in Principia Mathematica (Volume III, page 18):
  "The name ordinal numbers is commonly confined to the relation-numbers of well-ordered series [...] the relation-numbers of series in general are commonly called order types" (Whitehead & Russell, 1925)

- Associated with every order type an ordinal number (canonical representation of of the order type)
- Ordinal number is a set ordinally isomorphic to the order type class (Enderton, 1977)

Beyond infinity (transfinite ordinals)

$\omega = \{0,1,2,\ldots\}$
$\omega + 1 = \omega \cup \{\omega\} = \{0,1,2,\ldots,\omega\}$
$\omega + 2 = \omega+1 + 1 = \omega+1 \cup \{\omega+1\} = \{0,1,2,\ldots,\omega,\omega+1\}$
…
$\omega + \omega = \omega \cup \{\omega\} \cup \{\omega+1\} \cup \{\omega+2\} \ldots = \{0,1,2,\ldots,\omega,\omega+1,\ldots\} = \omega.2$
$\omega.2 + 1 = \{0,1,2,\ldots,\omega,\omega+1,\omega+2,\ldots,\omega.2\}$
…
$\omega.2 + \omega = \{0,1,2,\ldots,\omega,\omega+1,\omega+2,\ldots,\omega.2,\omega.2+1,\ldots\} = \omega.3$
…
$\omega.\omega = \{0,1,2,\ldots,\omega,\ldots,\omega.2,\ldots,\omega.3,\ldots\} = \omega^2$
$\omega^2 + 1 = \omega^2 \cup \{\omega\} = \{0,1,2,\ldots,\omega,\ldots,\omega.2,\ldots,\omega.3,\ldots,\omega^2\}$
…
ω^3, … ,ω^4, … , $\omega\hat{}\omega$, … , … , $\omega\hat{}\omega\hat{}\omega$, … , $\varepsilon_0 = \omega\hat{}\omega\hat{}\omega\hat{}\ldots$
ε_0+1, ε_0+2, …

Beyond infinity (transfinite ordinals)

- We can always get a bigger ordinal (Conway and Guy, 1996)
- So far all ordinals are countable (cardinality = $\aleph_0$)
- ω_1 first uncountable ordinal (cardinality = $\aleph_1$) is the set of all countable cardinals

Ordinals vs Cardinals

- Ordinals measure the *length of well-ordered structure*
- Cardinals measure the *size of set regardless of structure*
- Example:

  $\omega < \omega+1$

  $\omega \approx \omega+1 \approx \aleph_0$

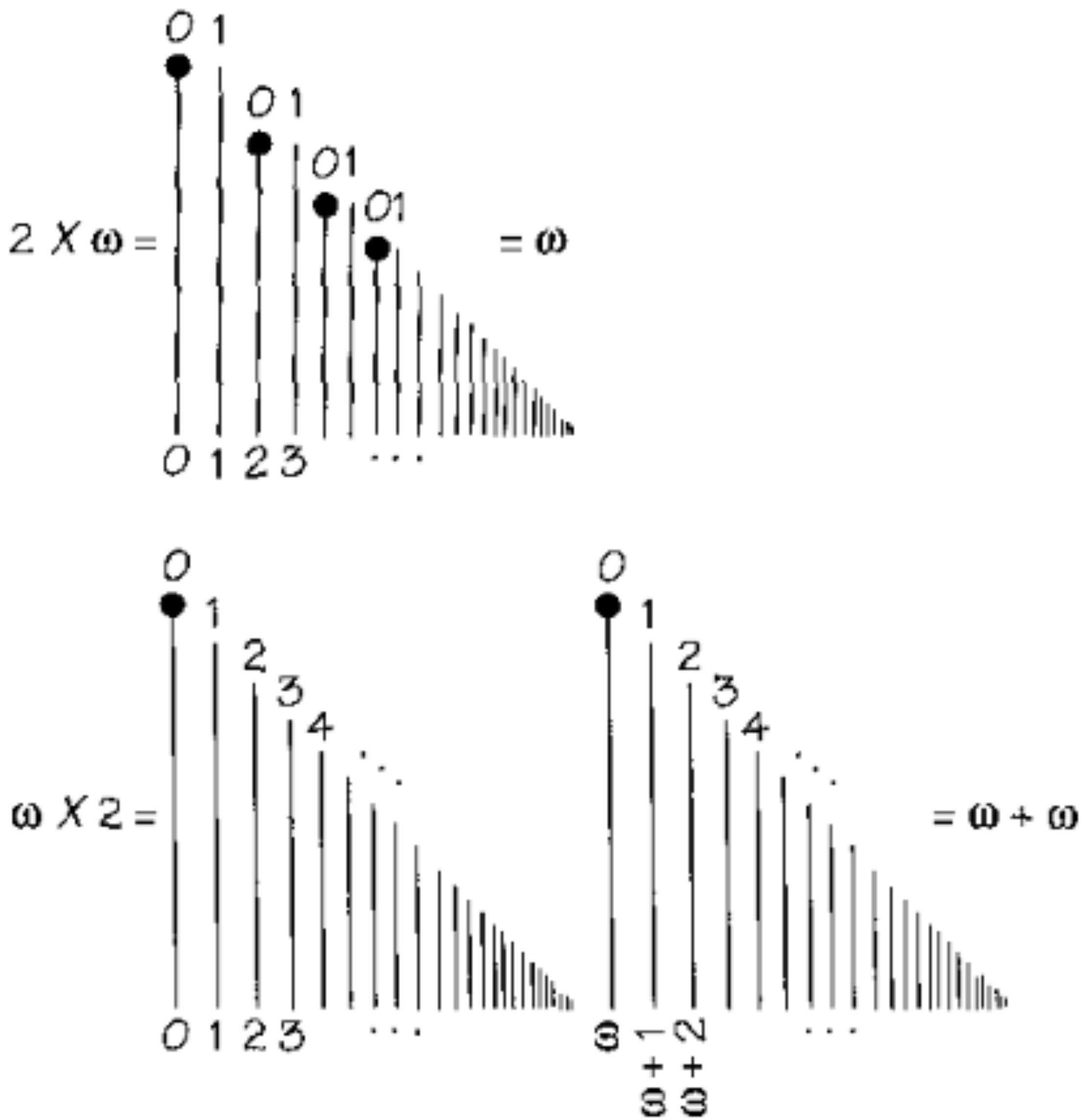
Ordinal Arithmetic

- Associative but not commutative (Lotfallah, 2007)

- Addition example:
  $\omega+1 \neq 1+\omega$
- Why?
  $\omega+1 = \{0,1,2,...,\omega\}$
  $1+\omega = \{1,0,1,2,...\} = \omega$

- Multiplication example:
  $\omega.2 \neq 2.\omega$
- Why?
  $\omega.2 = \omega = \omega+\omega = \{0,1,2,...,\omega,\omega+1,...\}$
  $2.\omega = \{0,1\}\times\omega = \{0,1, 0,1 ,... \omega \text{ times}\} = \omega$

Ordinal Arithmetic



ω.2 ≠ 2.ω from (Conway and Guy, 1996)

Ordinal Arithmetic (Definition)

- Defined by recursion (Lotfallah, 2007)
- Addition $\alpha+\beta$ =
  - base ($\beta = 0$): $\alpha+\beta = \alpha+0 = \alpha$
  - succ ($\beta = \gamma+1$): $\alpha+\beta = \alpha+\gamma+1 = (\alpha+\gamma) \cup \{\alpha+\gamma\}$
  - limit ($\beta = \sup\{\gamma:\gamma<\beta\}$): $\alpha+\beta = \sup\{\alpha+\gamma:\gamma<\beta\}$
- Multiplication $\alpha.\beta$ =
  - base ($\beta = 0$): $\alpha.\beta = \alpha.0 = 0$
  - succ ($\beta = \gamma+1$): $\alpha.\beta = \alpha.(\gamma+1) = (\alpha.\gamma)+\alpha$
  - limit ($\beta = \sup\{\gamma:\gamma<\beta\}$): $\alpha.\beta = \sup\{\alpha.\gamma:\gamma<\beta\}$
- Exponentiation α^β =
  - base ($\beta = 0$): $\alpha^\beta = \alpha^0 = 1$
  - succ ($\beta = \gamma+1$): $\alpha^\beta = \alpha^{(\gamma+1)} = \alpha^\gamma.\alpha$
  - limit ($\beta = \sup\{\gamma:\gamma<\beta\}$): $\alpha^\beta = \sup\{\alpha^\gamma:\gamma<\beta\}$

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