A proof of the Schroeder-Bernstein Theorem

CAS 701: Logic and Discrete Mathematics in Software Engineering

Christopher Kulbakas

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Department of Computing and Software
McMaster University

Background

- Motivation
  - How do you compare sizes of all sets (finite or infinite)?
  - Well, what if it is difficult to exhibit this bijection?
  - Also, is it possible to have an order relation on the cardinal numbers?
- Theorem Statement

The Schroeder-Bernstein Theorem. *For any sets A, B*

$$\left.\begin{array}{c}\exists\ f : A \to B \mid f \textit{ is a total, injective function} \\ \wedge \\ \exists\ g : B \to A, \mid g \textit{ is a total, injective function}\end{array}\right\} \Rightarrow \exists\ h : A \to B \mid h \textit{ is a bijection}$$

Note: given that the hypothesis holds, this theorem concludes that a bijection must exist, but does not produce this bijection

- Alternative Theorem Statement Form (just uses defined symbols $\succcurlyeq, \approx, |\cdot|$)

The Schroeder-Bernstein Theorem. *For any sets A, B*

$$|A| \succcurlyeq |B| \wedge |B| \succcurlyeq |A| \Rightarrow |A| \approx |B|$$

- This theorem is the "missing piece" (antisymmetry) in the definition of partial order applied to the cardinality of sets (reflexivity, transitivity are trivial)
- The value of this theorem lies in its application to infinite sets (result is trivial for finite sets since the hypothesis forces the sets A and B to have the same number of elements)
- In some people's opinion "this theorem is one of the first significant results in set theory"
- Other names include Cantor-Bernstein, Cantor-Schroeder-Bernstein
  - Name reflects authors: Georg Cantor, Felix Bernstein, Ernst Schroder
  - Bernstein (incorrectly) (1898) proved this theorem
  - Cantor (incorrectly) proved (1897) this theorem using the Axiom of Choice
  - Schroeder proved this theorem (1898) without relying on the Axiom of Choice
- Multiple proof strategies and multiple proofs using each strategy exist:
  - The proof in this presentation uses the Fixed Point Theorem
    - non-constructive (the proof does not show you how you can actually construct a bijection h)
    - reference: G. Hardegree, "Set Theory", Chapter 5, pages 10-11, http://people.umass.edu/gmhwww/595/text.htm

Helper Results

Fixed Point Theorem. [1] *For any set A, for any total function $f : \mathcal{P}(A) \to \mathcal{P}(A)$*

$$(\forall X, Y \mid X, Y \in \mathcal{P}(A) : (X \subseteq Y \Rightarrow f(X) \subseteq f(Y)) \Rightarrow (\exists Z \mid Z \subseteq A : f(Z) = Z)$$

Lemma 1. *For any sets A, B, for any paritions $(A_1, A_2), (B_1, B_2)$ of A, B, respectively*

$$\left.\begin{array}{c} \exists\ f_1 : A_1 \to B_1 \mid f_1 \text{ is a bijection} \\ \wedge \\ \exists\ f_2 : A_2 \to B_2, \mid f_2 \text{ is a bijection} \end{array}\right\} \Rightarrow \exists\ h : A \to B \mid h \text{ is a bijection}$$

Proof. Define $h : A \to B$ as follows: if $x \in A_1 \Rightarrow h(x) := f_1(x)$ and if $x \in A_2 \Rightarrow h(x) := f_2(x)$. Clearly, h is a bijection. □

Lemma 2. *For any set A, for any $X, Y \subseteq A$*

$$X \subseteq Y \Rightarrow A - Y \subseteq A - X$$

Proof. Let $X \subseteq Y$. If $x \in A - Y \Rightarrow x \in A \wedge \neg(x \in Y) \Rightarrow x \in A \wedge \neg(x \in X)(\because X \subseteq Y) \Rightarrow x \in A - X$. □

Lemma 3. *For any set A, for any $X, Y \subseteq A$, for any total function $f : A \to A$*

$$X \subseteq Y \Rightarrow \mathrm{R}_f(X) \subseteq \mathrm{R}_f(Y)$$

Proof. Let $X \subseteq Y$. Let $b \in \mathrm{R}_f(X) \Rightarrow (\exists a | a \in X : f(a) = b) \Rightarrow (\exists a | a \in Y : f(a) = b)(\because X \subseteq Y) \Rightarrow b \in R_f(Y)$. □

Corollary 1. *For any sets A, B, for any total functions $f : A \to B, g : B \to A$, for any $X, Y \subseteq A$*

$$X \subseteq Y \Rightarrow \mathrm{R}_g(B - \mathrm{R}_f(A - X)) \subseteq \mathrm{R}_g(B - \mathrm{R}_f(A - Y))$$

Proof. Assume $X \subseteq Y$
$\Rightarrow A - Y \subseteq A - X$ (by Lemma 2)
$\Rightarrow \mathrm{R}_f(A - Y) \subseteq \mathrm{R}_f(A - X)$ (by Lemma 3)
$\Rightarrow B - \mathrm{R}_f(A - X) \subseteq B - \mathrm{R}_f(A - Y)$ (by Lemma 2)
$\Rightarrow \mathrm{R}_g(B - \mathrm{R}_f(A - X)) \subseteq \mathrm{R}_g(B - \mathrm{R}_f(A - Y))$ (by Lemma 3) □

Lemma 4. *For any sets A, B, for any total function $f : A \to B$*

$$f \text{ is injective } \wedge X \subseteq \mathrm{D}_f \Rightarrow \exists h : X \to \mathrm{R}_f(X) \mid h \text{ is a bijection}$$

Proof. Define $h : X \to \mathrm{R}_f(X)$ as follows: $(\forall x | x \in X : h(x) := f(x))$. f total $\wedge X \subseteq D_f \Rightarrow h$ total. Since f is injective function, thus h is injective function. Lastly, let $b \in \mathrm{R}_f(X) \Rightarrow (\exists a | a \in X : f(a) = b) \Rightarrow (\exists a | a \in X : h(a) = b) \Rightarrow h$ is surjective. □

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[1] see reference for proof

A Proof

The Schroeder-Bernstein Theorem. *For any sets A, B*

$$\left.\begin{array}{c}\exists\, f : A \to B \mid f \textit{ is a total, injective function} \\ \wedge \\ \exists\, g : B \to A, \mid g \textit{ is a total, injective function}\end{array}\right\} \Rightarrow \exists\, h : A \to B \mid h \textit{ is a bijection}$$

Proof.

- Assume the hypothesis.

- Construct $F : \mathcal{P}(A) \to \mathcal{P}(A)$ where

$$F(X) := \mathrm{R}_g(B - \mathrm{R}_f(A - X))$$

Note: F is a function ($\because X = Y \Rightarrow \mathrm{R}_g(B - \mathrm{R}_f(A - X)) = \mathrm{R}_g(B - \mathrm{R}_f(A - Y)))$.

Note: F is total ($\because \forall X \in \mathcal{P}(A)\ \exists \mathrm{R}_g(B - \mathrm{R}_f(A - X))) \in \mathcal{P}(A))$.

- Choose any $X_1, X_2 \in \mathcal{P}(A)$ such that $X_1 \subseteq X_2$

  $\Rightarrow F(X_1) = \mathrm{R}_g(B - \mathrm{R}_f(A - X_1)) \subseteq \mathrm{R}_g(B - \mathrm{R}_f(A - X_2)) = F(X_2)$ (by Corollary 1)

  $\Rightarrow \exists Z \in \mathcal{P}(A) \mid F(Z) = \mathrm{R}_g(B - \mathrm{R}_f(A - Z)) = Z$ (by F.P.T.).

- Since f total, injective function and $A - Z \subseteq \mathrm{D}_f$ ($\because A - Z \subseteq A \wedge f$ total)

  $\Rightarrow \exists$ bijection $h_1 : A - Z \to \mathrm{R}_f(A - Z)$ (by Lemma 4).

- Since g total, injective function and $B - \mathrm{R}_f(A - Z) \subseteq \mathrm{D}_g$ ($\because B - \mathrm{R}_f(A - Z) \subseteq B \wedge g$ total)

  $\Rightarrow \exists$ bijection $h_2 : B - \mathrm{R}_f(A - Z) \to \mathrm{R}_g(B - \mathrm{R}_f(A - Z))$ (by Lemma 4)

  $\Rightarrow \exists$ bijection $h_2 : B - \mathrm{R}_f(A - Z) \to Z$ ($\because \mathrm{R}_g(B - \mathrm{R}_f(A - Z)) = Z$).

- Lastly,

  Since $(A - Z, Z)$ is a partition of A

  and $(B - \mathrm{R}_f(A - Z), \mathrm{R}_f(A - Z))$ is a partition of B

  $\Rightarrow \exists h : A \to B \mid h$ is a bijection (by Lemma 1)

$\square$