A Sound a Complete Semantic Tableau System for Propositional Logic

Valentín Cassano

Department of Computing and Software
McMaster University

November 20, 2008

Foreword on trees

Definition ((ordered) Tree)

A tree $\tau = (N, l, o, R)$ in which

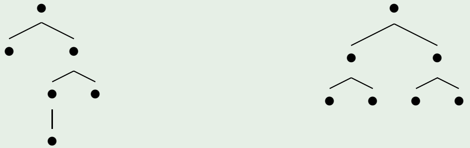
- N is a set of nodes
- $l : N \longrightarrow \mathbb{N}$ assigns to each node its level in the tree
- $R \subseteq (N \times N)$ is a relation on N such that
  - there is a unique node at level 1 (the root of the tree)
  - every node other than the root has a unique parent
  - for any node if $n_p R n_c$ then $l(n_c) = l(n_p) + 1$
- $o : N \longrightarrow \mathbb{N}$ assigns an order to the children of a node

Common terminology

- a *leaf* is a node that has no children
- a *simple node* is a node that has only one child (otherwise it is called a *junction point*)
- a *path* is a finite (or denumerable) sequence of nodes as usual (a branch is a path that ends in a leaf)

Foreword on trees

Example



Propositions and connectives

Definition

The language of propositional logic consists of

- proposition symbols: $p_0, p_1, \ldots$
- connectives: $\vee, \wedge, \rightarrow, \neg, \bot$
- auxiliary symbols: (,)

The set of propositions *PROP* is the smallest set X such that

- $p_i \in X$ and $\bot \in X$
- $\phi, \psi \in X$ then $(\phi \Box \psi) \in X$ where $\Box \in \{\vee, \wedge, \rightarrow\}$
- $\phi \in X$ then $(\neg\phi)$

Definition (Sub formulas)

The set S of sub formulas of a formula ϕ is defined by

$$\begin{aligned} sub(\phi) &= \{\phi\} \\ sub((\phi \Box \psi)) &= sub(\phi) \cup sub(\psi) \cup \{(\phi \Box \psi)\} \\ sub((\neg\phi)) &= sub(\phi) \cup \{(\neg\phi)\} \end{aligned}$$

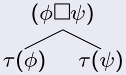
Propositions and connectives

Definition (Formation tree)

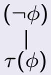
Let $\phi \in PROP$, then the formation tree τ for ϕ is defined as

$\tau(\phi) = \phi$

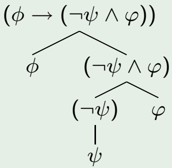
$\tau((\phi\Box\psi)) =$

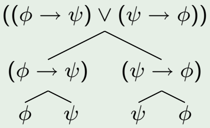


$\tau((\neg\phi)) =$



Example





Semantics

Definition

- ϕ is a *tautology* iff $[\![\phi]\!]_v = \textit{true}$ for all v
- ϕ is *satisfiable* iff $[\![\phi]\!]_v = \textit{true}$ for some v

Theorem

A formula ϕ is tautology iff $(\neg\phi)$ is unsatisfiable.

Definition (Truth set)

Let $[\![]\!]_v$ be a valuation function, and let Γ be the set of formulas that are *true* under v. Then the following holds

- exactly one of ϕ or $(\neg\phi)$ belongs to Γ
- $(\phi \wedge \psi) \in \Gamma$ iff $\phi \in \Gamma$ and $\psi \in \Gamma$
- $(\phi \vee \psi) \in \Gamma$ iff $\phi \in \Gamma$ or $\psi \in \Gamma$
- $(\phi \rightarrow \psi) \in \Gamma$ iff $\phi \notin \Gamma$ or $\psi \in \Gamma$

Theorem

A formula ϕ is satisfiable iff is is the member of some truth set Γ

Analytic Tableau

Theorem

For all $[\![]\!]_v$ *the following is true*

1. *if* $[\![(\neg\phi)]\!]_v = \mathit{true}$*, then* $[\![\phi]\!]_v = \mathit{false}$
2. *if* $[\![\phi]\!]_v = \mathit{true}$*, then* $[\![(\neg\phi)]\!]_v = \mathit{false}$
3. *if* $[\![(\phi \wedge \psi)]\!]_v = \mathit{true}$*, then* $[\![\phi]\!]_v = \mathit{true}$ *and* $[\![\psi]\!]_v = \mathit{true}$
4. *if* $[\![(\phi \wedge \psi)]\!]_v = \mathit{false}$*, then* $[\![\phi]\!]_v = \mathit{false}$ *or* $[\![\psi]\!]_v = \mathit{false}$
5. *if* $[\![(\phi \vee \psi)]\!]_v = \mathit{true}$*, then* $[\![\phi]\!]_v = \mathit{true}$ *or* $[\![\psi]\!]_v = \mathit{true}$
6. *if* $[\![(\phi \vee \psi)]\!]_v = \mathit{false}$*, then* $[\![\phi]\!]_v = \mathit{false}$ *and* $[\![\psi]\!]_v = \mathit{false}$
7. *if* $[\![(\phi \rightarrow \psi)]\!]_v = \mathit{true}$*, then* $[\![\phi]\!]_v = \mathit{false}$ *or* $[\![\psi]\!]_v = \mathit{true}$
8. *if* $[\![(\phi \rightarrow \psi)]\!]_v = \mathit{false}$*, then* $[\![\phi]\!]_v = \mathit{true}$ *and* $[\![\psi]\!]_v = \mathit{false}$

Definition (Signed formulas)

Let us introduce two symbols T and F, then under any interpretation $T[\phi]$ iff $[\![\phi]\!] = \mathit{true}$, and $F[\phi]$ iff $[\![\phi]\!] = \mathit{false}$. By the *conjugate* of T we mean F (and vice versa).

Tableau Method

Example

$$((\psi \to \psi) \wedge (\psi \to \varphi)) \to (\phi \to \varphi)$$

$$F[((\psi \to \psi) \wedge (\psi \to \varphi)) \to (\phi \to \varphi)]$$
$$T[(\psi \to \psi) \wedge (\psi \to \varphi)]$$
$$F[(\phi \to \varphi)]$$
$$T[(\psi \to \psi)]$$
$$T[(\psi \to \varphi)]$$
$$T[\phi]$$
$$F[\varphi]$$

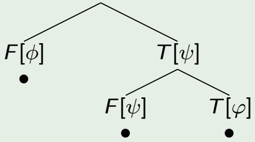


Tableau Method

Definition (Rules for the construction of the tableau)

(1) $\dfrac{T[(\neg\phi)]}{F[\phi]}$ (2) $\dfrac{F[(\neg\phi)]}{T[\phi]}$

(3) $\dfrac{T[(\phi \wedge \psi)]}{\begin{array}{c} T[\phi] \\ T[\psi] \end{array}}$ (4) $\dfrac{F[(\phi \wedge \psi)]}{F[\phi] | F[\psi]}$

(5) $\dfrac{F[(\phi \vee \psi)]}{\begin{array}{c} F[\phi] \\ F[\psi] \end{array}}$ (6) $\dfrac{T[(\phi \vee \psi)]}{T[\phi] | T[\psi]}$

(7) $\dfrac{F[(\phi \rightarrow \psi)]}{\begin{array}{c} T[\phi] \\ F[\psi] \end{array}}$ (8) $\dfrac{T[(\phi \rightarrow \psi)]}{F[\phi] | T[\psi]}$

Tableau Method

Theorem (Truth set)

Let Γ be a set of signed formulas, then Γ is a truth set iff

1. *Exactly one of $T[\phi]$ or $F[\phi]$ belongs to Γ*
2. *($\alpha \in \Gamma$ iff $\alpha_1 \in \Gamma$ and $\alpha_2 \in \Gamma$) and ($\beta \in \Gamma$ iff $\beta_1 \in \Gamma$ or $\beta_2 \in \Gamma$)*

Definition (Analytic Tableau)

An analytic tableau τ for a formula ϕ is an ordered binary tree whose nodes $n \in sub(\phi)$, and it is constructed as follows

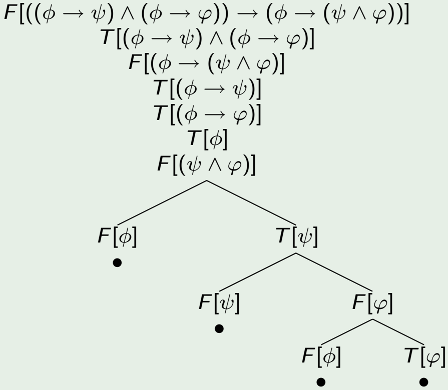
- the root of τ is $F[\phi]$
- Let τ be a tableau for $F[\phi]$ and θ a branch of τ, then we can extend τ by
  - if some α appears in θ, then we extend θ with $\frac{\alpha_1}{\alpha_2}$
  - if some β appears in θ, then we extend θ with $\beta_1 | \beta_2$

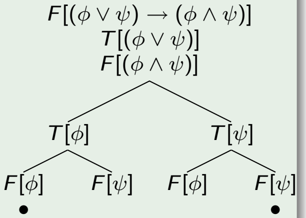
Definition

- A branch θ of τ is *closed* if some signed formula and its conjugate appears in θ
- A tableau τ is *closed* if every branch of τ is closed
- By a *proof* for ϕ we mean a closed tableau with $F[\phi]$ as its root

Tableau Method

Example





Completeness

Theorem (Consistency)

Let τ be any closed tableau, then its root is unsatisfiable

Definition

A branch θ of τ is complete if for every $\alpha \in \theta$, then $\alpha_1, \alpha_2 \in \theta$, and for every $\beta \in \theta$, either $\beta_1, \beta_2 \in \theta$. A tableau τ is *completed* if every branch is either closed or complete.

Theorem (Completeness)

Any complete open branch of the tableau is satisfiable. The proof relies on **Hintikka's lemma**

Lemma (Hintikka)

Every downward saturated set Γ is satisfiable.

Conclusions

- a presentation of a(nother) sound a complete proof system for propositional logic
- useful as provide counterexamples that violates the formula checked
- the method is applied in model checking and satisfiability proofs
- widely used as a proof procedure for modal logics

Questions

Questions? $>$ **/dev/null**

Bibliography

D. van Dalen. *Logic and Structure*. Fourth edition, Springer, 2003.

R. Smullyan. *First Order Logic*. Dover, 1995.