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02 Proposition Logic

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What is Propositional Logic?

- Propositional logic is the study of the truth or falsehood of propositions or sentences constructed using truth-functional connectives.
  - Also called sentential logic.
  - Began with the work of the Stoic philosophers, particularly Chrysippus, in the late 3rd century BCE.
- Most other logics are extensions of propositional logic.
- Main applications:
  - Logical arguments.
  - Logical circuits (e.g., electronic circuits).

Syntax

- A language of proposition logic is a pair $L = (\mathcal{A}, \mathcal{B})$ where:
  - $\mathcal{A}$ is a set of constants called propositional symbols or propositional letters.
  - $\mathcal{B}$ is a set of 0-ary, unary, and binary constructors called propositional connectives.
- A formula of L is a string of symbols inductively defined by the following formation rules:
  1. Each $p \in \mathcal{A}$ is a formula of L.
  2. If $c_0, c_1, c_2 \in \mathcal{B}$ are 0-ary, unary, and binary, respectively, and A_1, A_2 are formulas of L, then c_0, $(c_1\ A_1)$, and $(A_1\ c_2\ A_2)$ are formulas of L.
- Common propositional connectives: T, F (0-ary); $\neg$ (unary); $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\mid$ (binary).

An Example Language

- Let $L_0 = (\mathcal{A}, \mathcal{B})$ be the propositional language where:

  $\mathcal{A} = \{p_0, p_1, p_2, \ldots\}$.
  $\mathcal{B} = \{\neg, \Rightarrow\}$.

- The following abbreviations are employed:

T	denotes	$(p_0 \Rightarrow p_0)$.
F	denotes	$(\neg \mathrm{T})$.
$(A \vee B)$	denotes	$((\neg A) \Rightarrow B)$.
$(A \wedge B)$	denotes	$(\neg((\neg A) \vee (\neg B)))$.
$(A \Leftrightarrow B)$	denotes	$((A \Rightarrow B) \wedge (B \Rightarrow A))$.
$(A \mid B)$	denotes	$(\neg(A \wedge B))$.

Meaning of Propositional Connectives

- Each *n*-ary propositional connective denotes an assigned *n*-ary truth function.
- Examples:

	$\top$
	T

	$\bot$
	F

p	$(\neg p)$
T	F
F	T

p	q	$(p \wedge q)$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$(p \vee q)$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$(p \Rightarrow q)$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$(p \Leftrightarrow q)$
T	T	T
T	F	F
F	T	F
F	F	T

p	q	$(p \mid q)$
T	T	F
T	F	T
F	T	T
F	F	T

Complete Sets of Propositional Connectives

- A set $\mathcal{C}$ of proposition connectives is complete if every truth function can be represented by a formula using only members of $\mathcal{C}$.
- Examples of complete sets of propositional connectives:
  - $\{\neg, \Rightarrow\}$.
  - $\{\neg, \wedge\}$.
  - $\{\neg, \vee\}$.
  - $\{|\}$.

Semantics

- Let $L = (\mathcal{A}, \mathcal{B})$ be a language of propositional logic, and for each $c \in \mathcal{B}$, let f_c be its assigned truth function.
- A model for L is an (interpretation) function I that assigns a truth value in $\{\mathrm{T}, \mathrm{F}\}$ to each $p \in \mathcal{A}$.
- The valuation function for I is the function V that maps formulas of L to $\{\mathrm{T}, \mathrm{F}\}$ and satisfies the following conditions:
  1. If $p \in \mathcal{A}$, then $V(p) = I(p)$.
  2. If $c \in \mathcal{B}$ is 0-ary, then $V(c) = f_c$.
  3. If $c \in \mathcal{B}$ is unary and A is a formula of L, then $V((c\ A)) = f_c(V(A))$.
  4. If $c \in \mathcal{B}$ is binary and A, B are formulas of L, then $V((A\ c\ B)) = f_c(V(A), V(B))$.

Truth Tables

- Truth tables can be used to analyze the meaning of propositional formulas.
- Example (Rule of Contraposition):

p	q	$((p \Rightarrow q) \Leftrightarrow ((\neg q) \Rightarrow (\neg p)))$					
T	T	T	T	F	T	F	T
T	F	F	T	T	F	F	T
F	T	T	T	F	T	T	T
F	F	T	T	T	T	T	T

- A propositional formula A is a tautology and is valid if all of the final entries in the truth table for A are T.
- A propositional formula A is satisfiable if some of the final entries in the truth table for A are T.
- The validity of propositional formulas can be decided with truth tables—hence propositional logic is decidable!

Laws of Propositional Logic

- The laws of propositional logic are fundamental laws of most other common logics.
- Examples:
  - Law of Double Negation.
  - Law of Excluded Middle.
  - Law of Contraposition.
  - De Morgan's Laws.
  - Associative, commutative, and distributive laws.
  - Idempotent, identity, domination, and absorption laws.

A Hilbert-Style Proof System

Let **H** be the following Hilbert-style proof system for L_0:

- The logical axioms of **H** are all formulas of L_0 that are instances of the following three schemas:

  A1: $(A \Rightarrow (B \Rightarrow A))$.
  A2: $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$.
  A3: $((\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A))$.

- The single rule of inference of **H** is modus ponens:

  MP: From A and $(A \Rightarrow B)$, infer B.

Metatheorems of Propositional Logic

- Deduction Theorem. $\Sigma \cup \{A\} \vdash_{\mathbf{H}} B$ implies $\Sigma \vdash_{\mathbf{H}} A \Rightarrow B$.
- Soundness Theorem. $\Sigma \vdash_{\mathbf{H}} A$ implies $\Sigma \models A$.
- Completeness Theorem. $\Sigma \models A$ implies $\Sigma \vdash_{\mathbf{H}} A$.
- Soundness and Completeness Theorem (second form). Σ is consistent in $\mathbf{H}$ iff Σ is satisfiable.
- Compactness Theorem. If Σ is finitely satisfiable, then Σ is satisfiable.