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04 Orders and Lattices

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Pre-Orders

- A pre-order on a set S is a binary relation $\leq$ on S that is:
  - Reflexive: $\forall x \,.\, x \leq x$.
  - Transitive: $\forall x, y, z \,.\, x \leq y \land y \leq z \Rightarrow x \leq z$.
- Example: $(F, \Rightarrow)$ is a pre-order where F is a set of formulas and $\Rightarrow$ is implication.
- A pre-order can have cycles.
- Every binary relation R on a set S can be extended to a pre-order on S by taking the reflexive and transitive closure of R.

Partial Orders

- A weak partial order on a set S is a binary relation $\leq$ on S that is:
  - Reflexive: $\forall x \,.\, x \leq x$.
  - Antisymmetric: $\forall x, y \,.\, x \leq y \wedge y \leq x \Rightarrow x = y$.
  - Transitive: $\forall x, y, z \,.\, x \leq y \wedge y \leq z \Rightarrow x \leq z$.
- A strict partial order on a set S is a binary relation $<$ on S that is:
  - Irreflexive: $\forall x \,.\, \neg(x < x)$.
  - Asymmetric: $\forall x, y \,.\, x < y \Rightarrow \neg(y < x)$.
  - Transitive: $\forall x, y, z \,.\, x < y \wedge y < z \Rightarrow x < z$.
- Examples: $(\mathcal{P}(S), \subseteq)$ and $(\mathcal{P}(S), \subset)$ are weak and strict partial orders.
- A partial order does not have cycles.
- Every pre-order can be interpreted as a partial order.

Total Orders

- A weak total order on a set S is a binary relation $\leq$ on S that is:
  - Antisymmetric: $\forall x, y \,.\, x \leq y \land y \leq x \Rightarrow x = y$.
  - Transitive: $\forall x, y, z \,.\, x \leq y \land y \leq z \Rightarrow x \leq z$.
  - Total: $\forall x, y \,.\, x \leq y \lor y \leq x$.
- A strict total order on a set S is a binary relation $<$ on S that is:
  - Irreflexive: $\forall x \,.\, \neg(x < x)$.
  - Asymmetric: $\forall x, y \,.\, x < y \Rightarrow \neg(y < x)$.
  - Transitive: $\forall x, y, z \,.\, x < y \land y < z \Rightarrow x < z$.
  - Trichotomous: $\forall x, y \,.\, x < y \lor y < x \lor x = y$.

  Examples: $(\mathbf{N}, \leq)$, $(\mathbf{Z}, \leq)$, $(\mathbf{Q}, \leq)$, and $(\mathbf{R}, \leq)$ are weak total orders.

Some Basic Order Definitions

- Let $(P, \leq)$ be a partial order and $S \subseteq P$.
- A maximal element [minimal element] of S is a $M \in S$ $[m \in S]$ such that $\neg(M < x)$ $[\neg(x < m)]$ for all $x \in S$.
- The maximum element or greatest element [minimum element or least element] of S, if it exists, is a $M \in S$ $[m \in S]$ such that $x \leq M$ $[m \leq x]$ for all $x \in S$.
- An upper bound [lower bound] of S is a $u \in P$ $[l \in P]$ such that $x \leq u$ $[l \leq x]$ for all $x \in S$.
- The least upper bound or supremum [greatest lower bound or infimum of S, if it exists, is a $U \in P$ $[L \in P]$ such that U is an upper bound of S and, if u is an upper bound of S, then $U \leq u$ [L is a lower bound of S and, if l is a lower bound of L, then $l \leq L$].
- A function $f : P \to P$ is monotone with respect to $\leq$ if, for all $a, b \in P$, $a \leq b$ implies $f(a) \leq f(b)$.

Well-Orders

- A well-order on a set S is a weak total order $\leq$ on S such that every nonempty subset of S has a minimum element with respect to $\leq$.
- Examples: $(\mathbf{N}, \leq)$ and $(\mathbf{O}, \leq)$ are well-orders.
- A well-order has no infinite strictly decreasing sequences.
- The proof technique of induction and definition technique of recursion can be applied with respect to a well-ordered set.

Lattices

- A lattice is partial order $(L, \leq)$ such that:
  1. Every pair a, b of elements of L has a least upper bound in L called the join of a and b (joins exist).
  2. Every pair a, b of elements of L has a greatest lower bound in L called the meet of a and b (meets exist).
- The minimum and maximum of a lattice, if they exist, are called the bottom denoted by 0 or $\perp$ and the top denoted by 1 or $\top$, respectively.
- Examples:
  - $(\mathcal{P}(S), \subseteq)$ is a lattice with a bottom and top.
  - $(\mathbf{N}, \leq)$ is a lattice with a bottom but no top.
  - $(\mathbf{N}, |)$, where $a \mid b$ means a divides b, is a lattice with a bottom and top.

Semilattices

- A semilattice is partial order $(S, \leq)$ such that either joins exist or meets exist.
  - It is a join-semilattice if joins exist.
  - It is a meet-semilattice if meets exist.
- Examples:
  - Any lattice is a semilattice.
  - Any tree can be viewed as a semilattice.

Complete Lattices

- A complete lattice is a partial order $(L, \leq)$ such that, for each $S \subseteq L$, S has a least upper bound and greatest lower bound in L.
- Examples:
  - $(\mathcal{P}(S), \subseteq)$ is a complete lattice.
  - $(\mathbf{R}(0,1), \leq)$ is not a complete lattice.
  - $(\mathbf{R}[0,1], \leq)$ is a complete lattice.
  - $(\mathbf{Q}[0,1], \leq)$ is not a complete lattice.
  - $(\mathbf{N}, \leq)$ is not a complete lattice.
  - $(\mathbf{N}, |)$ is a complete lattice.

Knaster-Tarski Fixed Point Theorem

- Theorem. Let $(L, \leq)$ be a complete lattice and $f : L \longrightarrow L$ be monotone with respect to $\leq$. Then there exists a fixed point of f, i.e., there exists an $a \in L$ such that $f(a) = a$. Moreover, $(F, \leq)$, where F is the set of fixed points of f, is a complete lattice.
- There are several other fixed point theorems related to the Knaster-Tarski theorem.
- Fixed point theorems can be used to define objects by recursion.