

CAS 701 Fall 2008

06 Equational Logic and Algebraic Reasoning

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What is Equational Logic?

- **Equational logic** is first-order logic restricted to languages with no predicate symbols except $=$.
- An **equational theory** is a theory $T = (L, \Gamma)$ of equational logic such that each $A \in \Gamma$ is a universal closure of an equation of L , i.e., a closed formula of the form

$$\forall x_1, \dots, x_n . s = t.$$

- **Universal algebra** is the study of the models of equational theories.
- **Example.** The usual theory of groups is an equational theory.

Term Algebras

- A **ground term** is a variable-free term.
- The **term algebra** of a language $L = (\mathcal{C}, \mathcal{F}, \mathcal{P})$ of FOL with $\mathcal{C} = \{c_1, \dots, c_m\} \neq \emptyset$ and $\mathcal{F} = \{f_1, \dots, f_n\}$ is the algebra

$$(D, c_1, \dots, c_m, f_1, \dots, f_n)$$

where D is the set of ground terms of L .

- A **term model** of a theory $T = (L, \Gamma)$ of FOL is a model of T constructed from the term algebra of L .

Initial Models

- An **initial model** of a theory $T = (L, \Gamma)$ of FOL is a model $M = (D, I)$ of T such that:
 1. M has **no junk**, i.e., for every $d \in D$, there is a ground term of L whose value in M is d .
 2. M has **no confusion**, i.e., for all ground terms s and t , $s = t$ is true in M iff $s = t$ is valid in T .
- **Theorem.** Every equational theory has a unique initial model.
 - ▶ The initial model is a term model whose domain elements are equivalence classes of ground terms.
- The **initial model semantics** is often used in software engineering and computer science for equational theories instead of **first-order semantics**.

Example: Natural Numbers

- Let $L = (\{0\}, \{S\}, \{=\})$ be a language of FOL where S is unary.
- Consider the theory $T = (L, \emptyset)$.
 - ▶ T_1 is an equational theory.
- The initial model of T is $(\{0, S(0), S(S(0)), \dots\}, 0, S)$.
 - ▶ Is the term algebra of L .
 - ▶ Represents the natural numbers.
- The other models of T have junk or confusion.
 - ▶ Include both finite and infinite models.
 - ▶ Some contain “infinite” numbers.

Specifications and Descriptions

- A **specification** of a system S can be formalized as a theory T in some logic.
 - ▶ Each model of T represents an **implementation** of S .
 - ▶ First-order logic and simple type theory are good for specifying systems.
- A **description** of a system S can be formalized as a categorical theory T in some logic.
 - ▶ The unique model of T represents S (up to isomorphism).
 - ▶ Equational logic with the initial model semantics and simple type theory are good for describing systems, but first-order logic is often not good.

Substitutions

- Let $L = (\mathcal{C}, \mathcal{F}, \mathcal{P})$ be a language of FOL and \mathcal{T} be the set of terms of L .
- A **substitution** of L is a total function $\sigma : \mathcal{V} \rightarrow \mathcal{T}$.
- The **application** of a substitution σ to an expression E of L , written $E\sigma$, is defined by recursion as follows:
 - ▶ If $x \in \mathcal{V}$, $x\sigma = \sigma(x)$.
 - ▶ If $c \in \mathcal{C}$, $c\sigma = c$.
 - ▶ If $f \in \mathcal{F}$ is n -ary and t_1, \dots, t_n are terms of L , then $f(t_1, \dots, t_n)\sigma = f(t_1\sigma, \dots, t_n\sigma)$.
 - ▶ If $p \in \mathcal{P}$ is n -ary and t_1, \dots, t_n are terms of L , then $p(t_1, \dots, t_n)\sigma = p(t_1\sigma, \dots, t_n\sigma)$.
 - ▶ If A and B are formulas of L , then $(\neg A)\sigma = \neg(A\sigma)$ and $(A \Rightarrow B)\sigma = (A\sigma \Rightarrow B\sigma)$.
 - ▶ If $x \in \mathcal{V}$ and A is a formula of L , then $(\forall x . A)\sigma = (\forall x . A\sigma')$ where $\sigma'(x) = x$ and $\sigma'(y) = \sigma(y)$ for all $y \neq x$.

Matching

- Let s and t be terms and A and B be formulas of a language L of FOL.
- s **matches** t if there is a substitution σ of L such that $s = t\sigma$. Similarly, A **matches** B if there is a substitution σ of L such that $A = B\sigma$.
- If s matches t , then s is an **instance** of the **pattern** t .
- Matching is used in many places in CS and SE, e.g., in **term rewriting**.

Unification

- Let s and t be terms and A and B be formulas of a language L of FOL.
- s and t **unify** if there is a substitution σ of L such that $s\sigma = t\sigma$. Similarly, A and B **unify** if there is a substitution σ of L such that $A\sigma = B\sigma$.
- Unification is solving equations by syntax alone.
- The unifying substitution is called a **unifier**.
- Unification is used in many places in CS and SE, e.g., in **resolution theorem proving** and **logic programming**.

Most General Unifiers

- A **most general unifier** of s and t is a unifier σ of s and t such that, if σ' is a unifier of s and t , then there is substitution τ such that, for all $x \in \mathcal{V}$,

$$x\sigma' = (x\sigma)\tau.$$

- **Theorem.** For every pair of terms s and t of L , if s and t are unifiable, there is a most general unifier of s and t that is unique up to a renaming of variables.
- The first algorithm to compute the most general unifier of two first-order terms was given by Herbrand in 1930.

Algebraic Reasoning

- Let L be a language of FOL and \mathcal{T} be the set of terms of L .
- Let \mathcal{R} be a set of functions $r : \mathcal{T} \rightarrow \mathcal{T}$ called **computation rules** of L .
 - ▶ $r \in \mathcal{R}$ is **sound** if $t = r(t)$ for all $t \in \mathcal{T}$ with $r(t) \downarrow$.
- A **computation in** \mathcal{R} is a finite sequence $C = \langle t_1, \dots, t_n \rangle$ of terms of L such that, for all i with $1 \leq i < n$, there is some $r \in \mathcal{R}$ such that $t_{i+1} = r(t_i)$.
- **Proposition.** Suppose each $r \in \mathcal{R}$ is sound. If $\langle t_1, \dots, t_n \rangle$ is a computation in \mathcal{R} , then $t_1 = t_n$.

Equational Reasoning

- In an equational theory $T = (L, \Gamma)$, the fundamental rules of inference are **substitution** and **replacement**:
 - ▶ If $T \models s = t$, then $T \models s\sigma = t\sigma$ for every substitution σ of L .
 - ▶ If $T \models s = t$, then $T \models u = u'$ where u' is obtained by replacing one occurrence of s in u by t .
- Problem: How do we choose the substitutions?

Term Rewriting Systems

- Let L be a language of FOL and \mathcal{T} be the set of terms of L .
- A **rewrite rule** of L is a directed equation of L , written $s \rightarrow t$, such that all the variables of t are contained in s .
- The rewrite rule $s \rightarrow t$ is **applied** to a term u as follows:
 1. An occurrence of a term v is found in u that matches s . Let σ be the matching substitution.
 2. A term u' is obtained by replacing the occurrence of v in u by $t\sigma$. u' is the result of the application.
- A **term rewriting system** of L is a set \mathcal{R} of rewrite rules of L .

Reduction Relation

- The **reduction relation** $\rightarrow_{\mathcal{R}} \subseteq \mathcal{T} \times \mathcal{T}$ is the smallest relation containing \mathcal{R} and closed under **substitution** and **replacement**:
 - ▶ If $s \rightarrow_{\mathcal{R}} t$, then $s\sigma \rightarrow_{\mathcal{R}} t\sigma$ for every substitution σ of L .
 - ▶ If $s \rightarrow_{\mathcal{R}} t$, then $u \rightarrow_{\mathcal{R}} u'$ where u' is obtained by replacing one occurrence of s in u by t .
- Related relations:
 - $\rightarrow_{\mathcal{R}}^*$: the reflexive-transitive closure of $\rightarrow_{\mathcal{R}}$.
 - $\leftrightarrow_{\mathcal{R}}$: the symmetric closure of $\rightarrow_{\mathcal{R}}$.
 - $\leftrightarrow_{\mathcal{R}}^*$: the reflexive-symmetric-transitive closure of $\rightarrow_{\mathcal{R}}$.

Soundness and Completeness

- Let $T = (L, \Gamma)$ be an equational theory and \mathcal{R} be a term rewriting system for L .
- \mathcal{R} is **sound** with respect to T if
$$s \rightarrow_{\mathcal{R}} t \text{ implies } T \models s = t.$$
- \mathcal{R} is **complete** with respect to T if
$$T \models s = t \text{ implies } s \leftrightarrow_{\mathcal{R}}^* t.$$
- **Proposition.** For each $s = t \in \Gamma$, assume all the variables of t are contained in s . Then $\mathcal{R} = \{s \rightarrow t \mid s = t \in \Gamma\}$ is a term rewrite system of L which is sound and complete with respect to T .

Norm Forms

- Let \mathcal{R} be a term rewriting system for L .
- A term s of L is **in normal form** relative to \mathcal{R} if there is no term t such that $s \rightarrow_{\mathcal{R}} t$.
 - ▶ That is, no subterm of s matches the left side of a rewrite rule in \mathcal{R} .
- t is a **normal form** of s relative to \mathcal{R} if $s \rightarrow_{\mathcal{R}}^* t$ and t is in normal form relative to \mathcal{R} .

Properties of Term Rewriting Systems

- Let \mathcal{R} be a term rewriting system for L .
- \mathcal{R} is **Church-Rosser** if, for all terms s, t of L , $s \leftrightarrow_{\mathcal{R}}^* t$ iff there exists some u such that $s \rightarrow_{\mathcal{R}}^* u$ and $t \rightarrow_{\mathcal{R}}^* u$.
- \mathcal{R} is **confluent** if, for all terms s, t, u of L , $u \rightarrow_{\mathcal{R}}^* s$ and $u \rightarrow_{\mathcal{R}}^* t$ implies there is some term v such that $s \rightarrow_{\mathcal{R}}^* v$ and $t \rightarrow_{\mathcal{R}}^* v$.
- \mathcal{R} is **noetherian** or **finitely terminating** if there is no infinite chain of reductions

$$s_1 \rightarrow_{\mathcal{R}} s_2 \rightarrow_{\mathcal{R}} s_3 \rightarrow_{\mathcal{R}} \cdots$$

Theorems of Term Rewriting Systems

- Let \mathcal{R} be a term rewriting system for L .
- **Theorem.** \mathcal{R} is Church-Rosser iff \mathcal{R} is confluent.
- **Proposition.** If \mathcal{R} is confluent, then the normal form of a term of L is unique when it exists.
- **Proposition.** If \mathcal{R} is finitely terminating, then every term of L has a normal form.
- **Theorem.** Let $T = (L, \Gamma)$ be an equational theory and \mathcal{R} be sound and complete with respect to T , finite, confluent, and finitely terminating. Then:
 1. Every term s of L has a unique normal form t relative to \mathcal{R} such that $T \models s = t$.
 2. It is decidable whether $T \models s = t$.

Knuth-Bendix Completion Algorithm

- Let $T = (L, \Gamma)$ be an equational theory such that Γ is finite.
- Given Γ and a **reduction order** as input, the **Knuth-Bendix completion algorithm** does one of the following:
 1. Returns a term rewriting system \mathcal{R} for L that is sound and complete with respect to T , finite, confluent, and finitely terminating.
 2. Terminates with failure.
 3. Loops without terminating.
- The algorithm is composed of two steps:
 1. Creation of an initial set of rules by orienting the members of Γ according to the reduction order.
 2. Derivation of additional rules using **critical pairs**.

Restricted Systems of First-Order Logic

- A **conditional equation** is a formula of the form

$$A \Rightarrow s = t.$$

- ▶ Are used as conditional rewrite rules.

- A **Horn clause** is a formula of the form

$$A_1 \wedge \dots \wedge A_n \Rightarrow B$$

where A_1, \dots, A_n, B are positive literals and $n \geq 0$.

- ▶ A Horn clause of the form B is called a **goal**.

- Computation in restricted systems:

Kind of Formulas

equations

conditional equations

Horn clauses

Kind of Computation

term rewriting

constraint programming

logic programming