CAS 734 Winter 2005

02 The Axiomatic Method

Instructor: W. M. Farmer

Revised: 11 January 2005

What is Mathematics?

The essence of mathematics is a process consisting of three intertwined activities:

1. **Model creation**: Mathematical models representing mathematical aspects of the world are created.

2. **Model exploration**: The models are explored by:

   a. Stating and proving conjectures.

   b. Performing calculations.

   c. Creating and studying visual representations.

3. **Model connection**: Models with similar structure are connected to each other to facilitate the creation and exploration of new models.

What is a Logic?

- Informally, a logic is a system of reasoning.

- Formally, a **logic** is a family of **formal languages** with:

  1. A common syntax.
  2. A common semantics.
  3. A notion of **logical consequence**.

- A logic may include a **proof system** for **proving** that a given formula is a logical consequence of a given set of formulas.

- Examples:

  – Propositional logic.
  – First-order logic.
  – Simple type theory (higher-order logic).
  – Zermelo-Fraenkel set theory.

Language Syntax

- A language defines a collection of **expressions** formed from:
  - **Variables**.
  - **Constants** (nonlogical constants).
  - **Constructors** (logical constants).

- Two kinds of expressions:
  - **Terms**: Denote objects or values.
  - **Formulas**: Make assertions about objects or values.

- Some languages have constructors that bind variables (e.g., $\forall$, $\exists$, λ, I, ϵ, $\{\ |\ \}$).

Language Semantics

- A **model** M for a language L is a pair (D, V) where:
  1. D is a set of values called the **domain** that includes the truth values t and f.
  2. V is a function from the expressions of L to D called the **valuation function**.

- M **satisfies** a formula A of L, written $M \models A$, if $V(A) = \mathsf{t}$.

- M **satisfies** a set Σ of formulas of L, written $M \models \Sigma$, if M satisfies each $A \in \Sigma$.

- Σ is **satisfiable** if there exists some model for L that satisfies Σ.

- A is **valid**, written $\models A$, if every model for L satisfies A.

- A is a **logical consequence** of Σ, written $\Sigma \models A$, if every model for L that satisfies Σ also satisfies A.

Hilbert-Style Proof System

- A **Hilbert-style proof system H** for a language L consists of:

  1. A set of formulas of L called **logical axioms**.
  2. A set of **rules of inference**.

- A **proof** of A from Σ in **H** is a finite sequence $B_1, \ldots, B_n$ of formulas of L with $B_n = A$ such that each B_i is either a logical axiom, a member of Σ, or follows from earlier B_j by one of the rules of inference.

- A is **provable** from Σ in **H**, written $\Sigma \vdash_{\mathbf{H}} A$, if there is a proof of A from Σ in **H**.

- A is a **theorem** in **H**, written $\vdash_{\mathbf{H}} A$, if A is provable from $\emptyset$ in **H**.

- Σ is **consistent** in **H** if not every formula is provable from Σ in **H**.

Kinds of Proof Systems

- Hilbert style.
- Symmetric sequent (Gentzen).
- Asymmetric sequent.
- Natural deduction (Quine, Fitch, Berry).
- Semantic tableaux (Beth, Hintikka).
- Resolution (J. Robinson).

Soundness and Completeness

- Let **P** be a proof system for a language L.

- **P** is **sound** if

$$\Sigma \vdash_{\mathbf{P}} A \quad \text{implies} \quad \Sigma \models A.$$

- **P** is **complete** if

$$\Sigma \models A \quad \text{implies} \quad \Sigma \vdash_{\mathbf{P}} A.$$

Theories

- A **(axiomatic) theory** is a pair $T = (L, \Gamma)$ where:
  1. L is a language (the **language** of T).
  2. Γ is a set of formulas of L (the **axioms** of T).

- M is a **model** of T, written $M \models T$, if $M \models \Gamma$.

- A is **valid** in T, written $T \models A$, if $\Gamma \models A$.

- A is a **theorem** of T in **P**, written $T \vdash_{\mathbf{P}} A$, if $\Gamma \vdash_{\mathbf{P}} A$.

- T is **satisfiable** if Γ is satisfiable.

- T is **consistent** in **P** if Γ is consistent in **P**.

Logical View of Mathematical Problem Solving

Problem: Is $T \models A$ true?

Solution: Either

1. A **proof** showing $T \vdash_{\mathbf{P}} A$ for some sound proof system **P** for the language of T or

2. A **counterexample** in the form of a model M of T such that $M \models \neg A$.

What is the Axiomatic Method?

1. A mathematical model is expressed as a **theory** in a logic.

2. New concepts are introduced by making **definitions**.

3. Assertions about the model are stated as **theorems** and proved from the axioms using the laws of the logic.

Notes:

- The axiomatic method is a method of **communication**, not a method of **discovery** (Lakatos).

- The axiomatic method can be used as a method of **organization** and a method of **certification**.

History (1)

- Euclid (325–265 BC) used the axiomatic method to present the mathematics known in his time in the **Elements**.
  - The axioms were considered truths.

- The development of **noneuclidean geometry** by Bolyai, Gauss, and Lobachevskii (early 1800s) showed that axioms may be considered as just assumptions.

- Whitehead and Russell formalized a major portion of mathematics in the **Principia Mathematica** (1910–13).

- Bourbaki (mid 1900s) used the axiomatic method to codify mathematics in the 30 volume **Eléments de mathématique**.

History (2)

- Jutting (1970s) used De Bruijn's Automath proof assistant to formalize and verify Landau's **Grundlagen der Analysis**.

- Several libraries of formalized mathematics have been developed since the late 1980s using interactive theorem provers: Coq, HOL, IMPS, Isabelle, Mizar, Nqthm/ACL2, Nuprl, PVS.

Example: Theory of Partial Order

- Language: A language of first-order logic with a binary predicate symbol $\leq$.
  - $a \leq b$ is intended to mean a is less than or equal to b.

- Axioms:
  - **Reflexivity**. $\forall x \,.\, x \leq x$.
  - **Transitivity**. $\forall x, y, z \,.\, (x \leq y \wedge y \leq z) \Rightarrow x \leq z$.
  - **Antisymmetry**. $\forall x, y \,.\, (x \leq y \wedge y \leq x) \Rightarrow x = y$.

- The theory has infinitely many nonisomorphic models.

Example: Peano Arithmetic

- Language: A language of second-order logic with a constant symbol 0 and unary function symbol S.
  - 0 is intended to represent the number zero.
  - S is intended to represent the successor function, i.e., $S(a)$ means $a + 1$.

- Axioms:
  - **0 has no predecessor**. $\forall x \,.\, \neg(0 = S(x))$.
  - **S is injective**. $\forall x, y \,.\, S(x) = S(y) \Rightarrow x = y$.
  - **Induction principle**. $\forall P \,.\, (P(0) \wedge \forall x \,.\, P(x) \Rightarrow P(S(x))) \Rightarrow \forall x \,.\, P(x)$.

- Second-order Peano arithmetic is **categorical**, i.e, it has exactly one model up to isomorphism.

Benefits of the Axiomatic Method

1. **Conceptual clarity**: inessential details are omitted.

2. **Generality**: theorems hold in all models.

3. **Dual purpose**: a theory can be viewed as:

   a. An abstract mathematical model.

   b. A specification of a collection of mathematical models.

Shortcomings of the Axiomatic Method

1. Used with logics that do not conform to mathematical practice.
   - Are **theory oriented** rather than **practice oriented**.
2. Does not reflect how mathematics is discovered (Lakatos).
   - For example, there is no place for **counterexamples**.
3. Does not provide the means to build complex theories in sophisticated ways from simpler theories.
   - No distinction between **conservative** and **nonconservative** extensions.
4. Assumes everything is done within one (big) theory.
   - No means for **connecting theories** so that they can be used together.
   - No means for **parallel development** and **perspective switching**.

Attributes of Practice-Oriented Logics

- **Undefined terms**: Terms may be nondenoting.
- **Support for functions**: Functions may be partial, higher-order, and specified by lambda-abstraction.
- **Higher-order quantification**: Quantification is allowed over functions, relations, sets, etc.
- **System for classifying terms by value**: Terms are classified according to their values (using sorts or types).
- **Definite and indefinite description**: Terms may be formed from definite and indefinite descriptions.
- **Support for reasoning about syntax**: Expressions can denote both semantic values and syntactic expressions.

The Lakatosian Insight

- Presented by **Imre Lakatos** in *Proofs and Refutations*, Cambridge University Press, 1976.

- Mathematical reasoning is dialectical.
  - Dialectic between a theory and it theorems.
  - Dialectic between a conjecture and its proof.

- New mathematics is discovered by analyzing the proofs of conjectures according to the **method of proof and refutations**.

- The definition-theorem-proof style of presentation hides the true nature of mathematics.

Method of Proofs and Refutations

1. Propose a **conjecture** (which may actually be false).

2. Formulate a **proof experiment** that reduces the conjecture to a set of **subconjectures** (lemmas).

3. Look for **local counterexamples** to the subconjectures.

4. If a counterexample is not a **global counterexample** to the conjecture, use it to improve the subconjecture.

5. If it is a global counterexample, use it to improve the conjecture.

6. Start the process over with the improved conjecture.

Methods for Improving Conjectures

1. **Monster barring**:
   - Modify some of the definitions so that the counterexample is eliminated.
   - Both the conjecture and the proof experiment are unchanged.

2. **Exception barring**:
   - Add a condition to the conjecture so that the counterexample is eliminated.
   - Both the conjecture and the proof experiment are changed.

3. **Lemma incorporation**:
   - Make the truth of the subconjecture a condition to the conjecture.
   - The conjecture is changed, but the proof experiment is not.

Little Theories Method

- A complex body of mathematics is represented as a **network of theories**.
  - Bigger theories are composed of smaller theories.
  - Theories are linked by **interpretations**.
  - Reasoning is distributed over the network.

- Benefits:
  - Mathematics can be developed in the right language at the right level of abstraction.
  - Emphasizes reuse: if A is a theorem of T, then A may be reused in any "instance" of T.
  - Enables perspective switching.
  - Enables parallel development.
  - Inconsistency can be isolated.

- Implemented in IMPS.

Conclusion

- For the purpose of formalized mathematics, the axiomatic method needs to be expanded to include:
  1. Practice-oriented logics.
  2. Lakatos' method of proofs and refutations.
  3. The little theories method.

- This **expanded axiomatic method** provides an effective framework for **discovering**, **organizing**, **communicating**, and **certifying** mathematical knowledge.