Computing and Software 734
Formalized Mathematics
Fall 2006

Exercise 5
20 pts.
Due 26 October 2006

Assigned: 5 October 2006
Revised: 5 October 2006

Do this exercise by yourself.

Formalize in IMPS the following definitions, lemmas, and proofs (written by Henk Barendregt):

Let $\mathbf{N}$ be the set of natural numbers and P be the predicate on $\mathbf{N}$ defined by

$$\forall m : \mathbf{N} \,.\, P(m) \equiv \exists n : \mathbf{N} \,.\, 0 < m \wedge m^2 = 2n^2.$$

Lemma 1 $\forall m : \boldsymbol{N} \,.\, P(m) \supset \exists m' : \boldsymbol{N} \,.\, (m' < m \wedge P(m'))$.

Proof Indeed suppose $0 < m$ and $m^2 = 2n^2$. It follows that m^2 is even, but then m must be even, as odds square to odds. So $m = 2k$ and we have $2n^2 = m^2 = 4k^2$ which implies $n^2 = 2k^2$. Since $0 < m$, if follows that $0 < m^2$, $0 < n^2$, and $0 < n$. Therefore $P(n)$ holds. Moreover, $n^2 < n^2 + n^2 = m^2$, so $n^2 < m^2$ and hence $n < m$. So we can take $m' = n$. □

Lemma 2 $\forall m, n : \boldsymbol{N} \,.\, m^2 = 2n^2 \supset m = n = 0$.

Proof By Lemma 1, $\forall m : \mathbf{N} \,.\, \neg P(m)$, since there are no infinite descending sequences of natural numbers. Now suppose $m^2 = 2n^2$ with $m \neq 0$. Then $0 < m$ and hence $P(m)$, which is a contradiction. Therefore, $m = 0$. But then also $n = 0$. □

Send the instructor an IMPS file (i.e., a file with a `.t` extension) that includes:

1. All the def-forms that you used.
2. Proofs for all the def-theorems in the file.

3. Comments of the form

   `;;;`[*comment*]

   expressing your experiences in doing the exercise. For example, put a comment after each important step in a proof.