# Computing and Software 734
# Formalized Mathematics
# Winter 2008

## Exercise 5
## 20 pts.
## Due 12 March 2008

Assigned: 4 February 2008
Revised: 7 March 2008

Do this exercise by yourself.

Formalize in IMPS the following definitions, lemmas, and proofs (written by Henk Barendregt):

Let **N** be the set of natural numbers and $P$ be the predicate on **N** defined by

$$\forall\, m : \mathbf{N} \,.\, P(m) \equiv \exists\, n : \mathbf{N} \,.\, 0 < m \wedge m^2 = 2n^2.$$

**Lemma 1** $\forall\, m : \boldsymbol{N} \,.\, P(m) \supset \exists\, m' : \boldsymbol{N} \,.\, (m' < m \wedge P(m'))$.

**Proof** Indeed suppose $0 < m$ and $m^2 = 2n^2$. It follows that $m^2$ is even, but then $m$ must be even, as odds square to odds. So $m = 2k$ and we have $2n^2 = m^2 = 4k^2$ which implies $n^2 = 2k^2$. Since $0 < m$, if follows that $0 < m^2$, $0 < n^2$, and $0 < n$. Therefore $P(n)$ holds. Moreover, $n^2 < n^2 + n^2 = m^2$, so $n^2 < m^2$ and hence $n < m$. So we can take $m' = n$. □

**Lemma 2** $\forall\, m, n : \boldsymbol{N} \,.\, m^2 = 2n^2 \supset m = n = 0$.

**Proof** By Lemma 1, $\forall\, m : \mathbf{N} \,.\, \neg P(m)$, since there are no infinite descending sequences of natural numbers. Now suppose $m^2 = 2n^2$ with $m \neq 0$. Then $0 < m$ and hence $P(m)$, which is a contradiction. Therefore, $m = 0$. But then also $n = 0$. □

Send the instructor an IMPS file (i.e., a file with a `.t` extension) that includes:

1. All the def-forms that you used.

2. Proofs for all the def-theorems in the file.

3. Comments of the form

   `;;;`[*comment*]

   expressing your experiences in doing the exercise. For example, put a comment after each important step in a proof.