Computing and Software 734
Formalized Mathematics
Winter 2008

Exercise 8
50 pts.
Due 2 April 2008

Assigned: 7 March 2008
Revised: 7 March 2008

Do this exercise by yourself.

Part 1

Starting from scratch, create an IMPS theory of Peano arithmetic called
`peano1` with the constants 0, S, $+$, and $*$ and the following axioms:

1. 0 is not a successor.

2. S is injective.

3. Induction.

4. The two axioms that specify $+$ recursively.

5. The two axioms that specify $*$ recursively.

Part 2

Prove the following theorems in `peano1`:

1. 0 is the additive identity.

2. $S(0)$ is the multiplicative identity.

3. $+$ is associative.

4. $*$ is associative.

5. $+$ is commutative.

6. $*$ is commutative.

You will probably have to prove a few additional intermediate lemmas.

Part 3

Create a compound macete that reduces any ground expression of `peano1` (i.e., any expression containing no variables and no constants other than 0, S, $+$, and $*$) to an expression of the form $S^m(0)$ (i.e., S applied to 0 m times). Give several examples expressed as theorems to show that it works.

Part 4

Create the following three interpretations:

1. An interpretation of `peano1` in `h-o-real-arithmetic`.

2. An interpretation of `commutative-monoid-theory` in the additive part of `peano1`.

3. An interpretation of `commutative-monoid-theory` in the multiplicative part of `peano1`.