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01 What is Formalized Mathematics?

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What is Formalized Mathematics?

- Formalized mathematics is mathematics that is expressed and developed within a formal logic.
  - What is mathematics?
  - What is a formal logic?
- Formalized mathematics is the basis for formal methods.
- Before the invention of the modern computer, formalized mathematics was of only theoretical interest.
  - Biggest precomputer development was Whitehead and Russell's Principia Mathematica (1910–1913).
- Today formalized mathematics is usually done using various kinds of mechanized mathematics systems, particularly interactive theorem proving systems.

What is Mathematics?

The essence of mathematics is a process consisting of three intertwined activities:

1. Model creation: Mathematical models representing mathematical aspects of the world are created.
2. Model exploration: The models are explored by:
   - Stating and proving conjectures.
   - Performing computations.
   - Creating and studying visual representations.
3. Model connection: Models with similar structure are connected to each other to facilitate the creation and exploration of new models.

What is a Logic?

- Informally, a logic is a system of reasoning.
- Formally, a logic is a family of formal languages with:
  1. A common syntax.
  2. A common semantics.
  3. A notion of logical consequence.
- A logic may include a proof system for proving that a given formula is a logical consequence of a given set of formulas.
- Examples:
  - Propositional logic.
  - First-order logic.
  - Simple type theory (higher-order logic).
  - Zermelo-Fraenkel (ZF) set theory.

Benefits

- Formalized mathematics offers greater rigor than conventional mathematics.
  - The language of a formal logic is more precise than conventional mathematical language.
  - Deduction and computation rules are purely mechanical.
- Formalized mathematics offers more opportunity for mechanical support than conventional mathematics.
  - Proofs and computations can be checked by machine.
  - Parts of the mathematical reasoning process can be mechanized.
  - Abstract mathematics can be reused in widely varied contexts.
  - Mathematical knowledge can be created, stored, searched, applied, and disseminated in digital form.

Costs

- Formalized mathematics produces an overwhelming amount of details.
  - Formalized mathematics is almost impossible to do using conventional technology (i.e., pencil and paper).
  - Sophisticated computer support is needed.
- Very little basic mathematics is readily accessible in a formalized form.
  - In applications of formalized mathematics, more effort is usually put into developing background mathematics than into solving the main problem.
- In order to capitalize on the benefits of formalized mathematics, new logics and mathematics development techniques are needed.

Misconception 1

Formalized mathematics is an all-or-nothing alternative to conventional mathematics.

- Formalized mathematics is intended to supplement, not replace, conventional mathematics.
- Formalized mathematics is not formal mathematics—which is an entirely formal, machine-oriented approach to mathematics.
- Formalized mathematics is human oriented and emphasizes the "mathematics" instead of the "formality".

Misconception 2

A formal proof system enables proof discovery to be automated.

- A formal proof system enables proof search to be automated.
- Automated proof search is generally not an effective way to find proofs.
  - The search space is too big.
  - Failed searches provide little useful feedback.
- Automated proof search is implemented by automated theorems provers like Otter.
- Proof discovery is a highly human activity that cannot be easily automated.

Misconception 3

A formal proof is almost an absolute guarantee of the correctness of the theorem it proves.

- The correctness of a formal proof can be checked with great efficiency and certainty.
- The correctness of the proof guarantees that the theorem is true, but it does not guarantee that the theorem says what it is intended to say.
- Conviction in mathematics is established by cross checks, not proofs.

Misconception 4

The definition/theorem/proof model of presentation in mathematics papers and textbooks faithfully reflects how mathematics is developed.

- As Imre Lakatos noted in his famous Proofs and Refutations (Cambridge University Press, 1976), the development of mathematics is dialectical, not sequential.
  - Dialectic between an assumption and its consequences.
  - Dialectic between a conjecture and its proof.
- New mathematics is discovered by analyzing the proofs of conjectures according to Lakatos' method of proof and refutations.
- The definition/theorem/proof style of presentation hides the true nature of mathematics.

Research Challenges in Formalized Mathematics

1. Develop more effective user interfaces for mechanized mathematics systems.
2. Integrate symbolic computation and formal deduction.
3. Design and implement practice-oriented logics that allow users to better follow standard mathematical practice.
4. Build mechanized mathematics systems that support a wide spectrum of the mathematics process.
5. Create a universal digital mathematics library.

Future of Formalized Mathematics

- Formalized mathematics has the potential to transform how mathematics is learned and practiced.
- The impact will probably be much greater in science and engineering and in mathematics education than in mathematics research.
- Availability of effective formalized mathematics could enable significant portions of the software development process to be mechanized.