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02 Review of Mathematical Logic

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What is Mathematical Logic?

- Study of the principles underlying mathematical reasoning.
  - Central idea: logical consequence.
- Branch of mathematics.
- Makes explicit several fundamental distinctions:
  - Syntax vs. semantics.
  - Language vs. metalanguage.
  - Theory vs. model.
  - Truth vs. proof.
- Principal tools: formal systems called logics.

Syntax vs. Semantics

- The syntax of a language is concerned with how the expressions of the language are constructed.
  - For example, "the numeral 144 has three digits" is a statement about syntax.
- The semantics of a language is concerned with what the expressions of the language mean.
  - For example, "the number 144 is a perfect square" is a statement about semantics.
- This distinction is crucial in mathematics and computing.
  - Confusion between syntax and semantics is the source of many errors.
- Logic carefully disentangles the roles of syntax and semantics in reasoning.

What is a Logic?

- Informally, a logic is a system of reasoning.
- Formally, a logic is a family of formal languages with:
  1. A common syntax.
  2. A common semantics.
  3. A notion of logical consequence.
- A logic may include a proof system for proving that a given formula is a logical consequence of a given set of formulas.
- Examples:
  - Propositional logic.
  - First-order logic.
  - Simple type theory (higher-order logic).
  - Zermelo-Fraenkel (ZF) set theory.

Language Syntax

- A language defines a collection of expressions formed from:
  - Variables.
  - Constants (nonlogical constants).
  - Constructors (logical constants).
- Three kinds of expressions:
  - Terms: Denote objects or values.
  - Formulas: Make assertions about objects or values.
  - Types: Restrict the scope of variables, control the formation of expressions, and classify expressions by their values.
- Some languages have constructors that bind variables (e.g., $\forall$, $\exists$, λ, ι, ϵ, $\{ \mid \}$).

Language Semantics

- A model M for a language L is a pair (D, V) where:
  1. D is a set of values called the domain that includes the truth values T and F.
  2. V is a function from the expressions of L to D called the valuation function.
- M satisfies a formula A of L, written $M \models A$, if $V(A) = \mathrm{T}$.
- M satisfies a set Σ of formulas of L, written $M \models \Sigma$, if M satisfies each $A \in \Sigma$.
- A is a semantic consequence of Σ, written $\Sigma \models A$, if every model for L that satisfies Σ also satisfies A.
- A is valid, written $\models A$, if every model for L satisfies A.
- Σ is satisfiable if there exists some model for L that satisfies Σ.

Language vs. Metalanguage

- A language is for talking about a certain subject.
- A metalanguage for a language L is a language for talking about L itself.
- A natural language, such as English, usually serves as its own metalanguage.
  - As a result, the distinction is not explicit in English.
- A formal language, such as a logical or programming language, usually is not expressive enough to serve as its own metalanguage.
  - A metalanguage of a formal language may be a formal language, but usually it is only informal.

Proof

- Mathematical proof is an essential component of the mathematics process which is unique to mathematics.
- It is a method of communication, certification, and discovery.
- An informal proof is a convincing argument that a statement about a mathematical model is true.
- A formal proof is a logical deduction from a set of premises to a conclusion.
  - Can be mechanically checked.
- A formal proof can be presented in two ways:
  - As a description of the actual deduction.
  - As a prescription for creating the deduction.

Hilbert-Style Proof Systems

- A Hilbert-style proof system **P** for a language L consists of:
  1. A set of formulas of L called logical axioms.
  2. A set of rules of inference.
- A proof of A from Σ in **P** is a finite sequence $B_1, \ldots, B_n$ of formulas of L with $B_n = A$ such that each B_i is either a logical axiom, a member of Σ, or follows from earlier B_j by one of the rules of inference.
- A is syntactic consequence of Σ in **P**, written $\Sigma \vdash_{\mathbf{P}} A$, if there is a proof of A from Σ in **P**.
- A is a theorem in **P**, written $\vdash_{\mathbf{P}} A$, if there is a proof of A from $\emptyset$ in **P**.
- Σ is consistent in **P** if not every formula is a syntactic consequence of Σ in **P**.

Kinds of Proof Systems

- Hilbert style.
- Symmetric sequent (Gentzen).
- Asymmetric sequent.
- Natural deduction (Gentzen, Quine, Fitch, Berry).
- Semantic tableaux (Beth, Hintikka).
- Resolution (J. Robinson).

Soundness and Completeness

- Let **P** be a proof system for a language L.
- **P** is sound if

$$\Sigma \vdash_{\mathbf{P}} A \;\text{ implies }\; \Sigma \models A.$$

- **P** is complete if

$$\Sigma \models A \;\text{ implies }\; \Sigma \vdash_{\mathbf{P}} A.$$

- A unsound proof system is not usually very useful, while a sound but incomplete proof system can be quite useful.

Axiomatic Theories

- An axiomatic theory is a pair $T = (L, \Gamma)$ where:
  1. L is a language (the language of T).
  2. Γ is a set of formulas of L (the axioms of T).
- M is a model of T, written $M \models T$, if $M \models \Gamma$.
- A is valid in T, written $T \models A$, if $\Gamma \models A$.
- A is a theorem of T in **P**, written $T \vdash_{\mathbf{P}} A$, if $\Gamma \vdash_{\mathbf{P}} A$.
- T is satisfiable if Γ is satisfiable.
- T is consistent in **P** if Γ is consistent in **P**.

Theory vs. Model

- A model for a language is a concrete mathematical model.
- A axiomatic theory is an abstract mathematical model.
- An axiomatic theory can be viewed as a specification of its models.
  - A theory is to a model as a specification is to an implementation.
- Axiomatic theories fall into two categories:
  - Those intended to describe a single model (e.g., a theory of natural number arithmetic).
  - Those intended to describe a collection of models (e.g., a theory of monoids).

Truth vs. Proof

Semantics	**Syntax**
truth	proof
semantic consequence	syntactic consequence
A is valid $\models A$	A is a theorem in $\mathbf{P}$ $\vdash_{\mathbf{P}} A$
A is valid in T $T \models A$	A is a theorem of T in $\mathbf{P}$ $T \vdash_{\mathbf{P}} A$
T is satisfiable	T is consistent in $\mathbf{P}$

- Semantic consequence and syntactic consequence are different forms of logical consequence.
- The semantic and syntactic notions are equivalent in the most common logics:
  - Propositional logic (Bernays, 1918).
  - First-order logic (Gödel, 1930).
  - Simple type theory (Henkin, 1950).

Mathematical Problems: Fundamental Form

- Most mathematical problems can be expressed as statements of the form

  $$T \models A$$

  where T is an axiomatic theory and A is a formula.
- There are three basic ways of deciding whether or not $T \models A$:
  1. Model checking:
     Show that $M \models A$ for each model M of T.
  2. Proof:
     Show $T \vdash_{\mathbf{P}} A$ for some sound proof system $\mathbf{P}$.
  3. Counterexample:
     Show $M \models \neg A$ for some model M of T.