CS 773 Winter 2001

02. The Axiomatic Method

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Revised: 18 January 2001

What is the Axiomatic Method?

1. A mathematical model is expressed as a set of axioms (in a language) called an **axiomatic theory**
2. New concepts are introduced by making **definitions**
3. Assertions about the model are stated as **theorems** and proved from the axioms

Notes:

- The axiomatic method is a method of **presentation**, not a method of **discovery** (Lakatos)
- The axiomatic method can be used as a method of **organization**

History

- Euclid (325–265 BC) used the axiomatic method to present the mathematics known in his time in the **Elements**
  - The axioms were considered truths
- The development of **noneuclidean geometry** by Bolyai, Gauss, and Lobachevskii (early 1800s) showed that axioms may be considered as just assumptions
- Whitehead and Russell formalized a major portion of mathematics in the **Principia Mathematica** (1910–1913)
- Bourbaki (mid 1900s) used the axiomatic method to codify mathematics in the 30 volume **Eléments de mathématique**
- Several libraries of formalized mathematics have been developed since the late 1980s using interactive theorem provers: HOL, IMPS, Isabelle, Mizar, Nqthm, Nuprl, PVS

Axiomatic Theories

- **Theory** = formal language + set of axioms
- **Language**: vocabulary for objects and their properties
  - Has a precise semantics (with a notion of logical consequence)
  - Can be used to describe multiple situations
  - The language usually belongs to a **logic**
- **Axioms**: assumptions about the objects and properties
  - Specify a class of **models**
  - Basis for proving **theorems**

Example: Theory of Partial Order

- Language: A first-order logic language with a 2-place predicate symbol $\leq$
  - $a \leq b$ is intended to mean a is less than or equal to b
- Axioms:
  - **Reflexivity**. $\forall x \,.\, x \leq x$
  - **Transitivity**. $\forall x, y, z \,.\, (x \leq y \wedge y \leq z) \supset x \leq z$
  - **Antisymmetry**. $\forall x, y \,.\, (x \leq y \wedge y \leq x) \supset x = y$
- The theory has infinitely many nonisomorphic models

Example: Peano Arithmetic

- Language: A second-order logic language with a constant symbol 0 and 1-place function symbol S
  - 0 is intended to represent the number zero
  - S is intended to represent the successor function, i.e., $S(a)$ means $a + 1$
- Axioms:
  - **0 has no predecessor**. $\forall x \,.\, \neg(0 = S(x))$
  - **S is injective**. $\forall x, y \,.\, S(x) = S(y) \supset x = y$
  - **Induction principle**. $\forall P \,.\, (P(0) \wedge \forall x \,.\, P(x) \supset P(S(x))) \supset \forall x \,.\, P(x)$
- Second-order Peano arithmetic is **categorical**, i.e, it has exactly one model up to isomorphism

Benefits of Axiomatic Theories

- **Conceptual clarity**: inessential details are omitted
- **Generality**: theorems hold in all models
- **Dual purpose**: a theory can be viewed as:
  - An abstract mathematical model
  - A specification of a collection of mathematical models

Theory Interpretations

- A **translation** Φ from T to T' is a function that maps the primitive symbols of T to expressions of T' satisfying certain syntactic conditions
- Φ determines:
  - A mapping of expressions of T to expressions of T'
  - Set of sentences called **obligations**
- Φ is an **interpretation** if it maps the theorems of T to theorems of T'
  - Sufficient condition: the obligations of Φ are theorems of T'
- **Interpretations are information conduits!**

Example: Theory of Computer Networks

- Theory name: Networks
- Language: Many-sorted first-order logic language with the following sorts and function symbols:

Sorts	**Function symbols**
boxes	box-of-interface
wires	wire-of-interface
interfaces	address-of-interface
addresses	

- Example axioms:
  - "Every box has a unique loopback interface"
  - "The address of a loopback interface is 127.0.0.1"

Example: Theory of Bipartite Graphs

- Theory name: Bipartite Graphs
- Language: Many-sorted first-order logic language with the following sorts and function symbols:

Sorts	**Operators**
red-nodes	red-node-of-edge
blue-nodes	blue-node-of-edge
edges	

- No explicit axioms

Example: Bipartite Graphs to Networks

- Let $\Phi_{BG \to N}$ be the translation from Bipartite Graphs to Networks defined by:

  red-nodes $\mapsto$ boxes
  blue-nodes $\mapsto$ wires
  edges $\mapsto$ interfaces
  red-node-of-edge $\mapsto$ box-of-interface
  blue-node-of-edge $\mapsto$ wire-of-interface

- $\Phi_{BG \to N}$ has no obligations
- $\Phi_{BG \to N}$ is an interpretation
  - "Transitivity of red-to-red connectivity" maps to "transitivity of box-to-box connectivity"

Example: Symmetry Interpretation

- Let $\Phi_{BG \to BG}$ be the translation from Bipartite Graphs to Bipartite Graphs defined by:

  red-nodes $\mapsto$ blue-nodes
  blue-nodes $\mapsto$ red-nodes
  edges $\mapsto$ edges
  red-node-of-edge $\mapsto$ blue-node-of-edge
  blue-node-of-edge $\mapsto$ red-node-of-edge

- $\Phi_{BG \to BG}$ has no obligations:
- $\Phi_{BG \to BG}$ is an interpretation
  - "Transitivity of red to red connectivity" maps to "transitivity of blue to blue connectivity"

Two Versions of the Axiomatic Method

1. **Big Theory**: A body of mathematics is entirely represented in one theory
   - Often a powerful, highly expressive theory like set theory is selected
   - All reasoning is performed within this single theory
2. **Little Theories**: A body of mathematics is represented as a network of theories
   - Bigger theories are composed of smaller theories
   - Theories are linked by interpretations
   - Reasoning is distributed over the network

Benefits of Little Theories

- Mathematics can be developed using the most appropriate vocabulary at the most appropriate level of abstraction
- Emphasizes reuse: if A is a theorem of T, then A may be reused in any "instance" of T
- Enables perspective switching
- Enables parallel development
- Inconsistency can be isolated: there are no interpretations of an inconsistent theory in a consistent theory, so inconsistency cannot spread from one theory to another