CS 773 Winter 2001

08. Set Theory

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What is Set Theory?

- Based on two simple notions:
  - Set
  - Membership
- Nearly all mathematical concepts can be expressed in terms of set and membership
- Set theory is, at least among mathematicians, the most popular foundation for mathematics
- There are many different formalizations of set theory

Formalizations of Set Theory

- The standard formalization of set theory is known as **Zermelo-Fraenkel (ZF) set theory** [Zermelo, 1908]
- Other major formalizations:
  - **von-Neumann-Bernays-Gödel (NBG) set theory** [von Neumann, 1925]
  - **Morse-Kelley (MK) set theory** [Kelley, 1955]
  - **Tarski-Grothendieck set theory** [Tarski, 1938]
  - **New Foundations (NF)** [Quine, 1937]

ZF

- Proposed by Zermelo in 1908
  - Developed to avoid the set-theoretic paradoxes
  - Improvements made by Fraenkel (1922) and Skolem (1923)
- ZF is formalized as a theory in first-order logic
  - Language contains two predicate symbols $=$ and $\in$
  - Is not finitely axiomatizable
- Proper classes (e.g., the collection of all sets) are not first-class objects
  - They cannot be denoted by terms
  - They are used in the metatheory
  - They can be denoted by predicate symbols
- ZF is an exceedingly rich theory

Axioms of ZF

1. Extensionality
2. Foundation
3. Comprehension scheme
4. Pairing
5. Union
6. Replacement scheme
7. Powerset
8. Infinity
9. Choice

NBG

- Proposed by von Neumann in 1925
  - Improvements made by R. Robinson (1937), Bernays (1937–54), and Gödel (1940)
- NBG is formalized as a theory in first-order logic
  - Has the same language as ZF
  - Is finitely axiomatizable
- Proper classes are first-class objects
- NBG is closely related to ZF
  - NBG is consistent iff ZF is consistent
  - NBG and ZF share the same intuitive model of the iterated hierarchy of sets
  - NBG and ZF have very similar axioms