**CS 773 Winter 2001**

# 09. Theory Development

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## Overview

- Mathematics is a process of creating, exploring, and connecting mathematical models
- Formalized mathematics is a process of developing axiomatic theories in a formal logic involving:
  - Theory creation
  - Conservative theory extension
  - Theory exploration
  - Theory interpretation



## Theory Creation

- Theories can be created in a several ways:
  - From scratch
  - By forming a union of a set of theories
  - By adding new vocabulary and axioms to a theory
  - By instantiating a parameterized theory
  - By instantiating a theory via an interpretation
- A theory may be required to contain a **kernel theory** which includes the machinery common to all theories



## Conservative Theory Extension

- A conservative extension $T'$ of $T$ adds new machinery to $T$ without compromising the original machinery of $T$
- The **obligation** of a purported conservative extension is a formula that implies that the extension is conservative
- Since $T$ and $T'$ are essentially the same theory, $T'$ can be implemented by overwriting $T$
  - Avoids a proliferation of closely related theories
- There are two important kinds of conservative extensions that add new vocabulary to a theory:
  - Definitions
  - Profiles

## Definitions

- A **definition** is a conservative extension that adds a new symbol $s$ and a defining axiom $A(s)$ to a theory $T$
  - In some logics, the defining axiom can have the form $s = D$ (where $s$ does not occur in $D$)
- The obligation of the definition is

$$\exists !x \,.\, A(x)$$

- The symbol $s$ can usually be eliminated from any new expression of involving $s$



## Profiles

- A **profile** is a conservative extension that adds a set $\{s_1, \ldots, s_n\}$ of symbols and a profiling axiom $A(s_1, \ldots, s_n)$ to a theory $T$
- The obligation of the profile is

$$\exists x_1, \ldots, x_n \,.\, A(x_1, \ldots, x_n)$$

- The symbols $s_1, \ldots, s_n$ cannot usually be eliminated from expressions involving $s_1, \ldots, s_n$
- Profiles can be used for introducing:
  - Underspecified objects
  - Recursively defined functions
  - Abstract datatypes



## Theory Exploration

- The logical consequences of a theory are explored by:
  - Proving conjectures
  - Performing computations
- Products of theory exploration:
  - Theorems
  - Proofs
  - Counterexamples
  - Computations
- Tools of theory exploration:
  - Theorems
  - Transformers



## Theorems

- Facts about a theory are recorded as theorems
- A theorem is usually installed in a theory only if it has been verified by a proof
- A theorem may sometimes be installed without a proof:
  - A theorem verified by a decision procedure
  - A theorem verified by a counterexample
  - A theorem imported via an interpretation
  - A theorem shown by a metatheorem

## Transformers

- A **transformer** is a function that maps the expressions of a language $L$ to the expressions of a language $L'$
  - Usually, $L \leq L'$, $L' \leq L$, or $L = L'$
- A transformer can be used to represent an expression transforming operation such as an evaluator, a simplifier, a rewrite rule, a rule of inference, a decision procedure, or an interpretation of one language in another
- Sound transformers can be:
  - Generated from theorems (e.g., theorem macetes)
  - Constructed from other transformers using certain constructors (e.g., compound macetes)
  - Obtained by instantiating abstract transformers (e.g., algebraic and order processors)
  - Manually defined and verified



## Theory Interpretations

- Theory interpretations can be used to:
  - Transport theorems, definitions, and profiles
  - Instantiate theories
  - Compare the strength of theories
  - Show relative consistency of theories
  - Show theory extension conservativity
- Logic interpretations can be used to interpret a theory in one logic in a theory of another logic