

Popular View

- The essence of mathematics is a huge body of concepts and facts about such things as time, measure, pattern, space, and logical consequence
- New concepts and facts are discovered by the definition-theorem-proof process
- Mathematics is infallible
 - Old concepts and facts are immutable
- Conjectures are either proved with a proof or refuted with a counterexample

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02. What is Mathematics?

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Hallmarks of Mathematics

1. Abstraction
2. Symbolic methods
3. Conditional reasoning
4. Proof
5. Rigor
 - (a) Unambiguous language
 - (b) No hidden assumptions
 - (c) Conclusions follow from assumptions
6. High (and often unexpected) applicability to the real world
7. Extremely long historical development

Mathematics-as-Process View

The essence of mathematics is a process consisting of three intertwined activities:

1. **Model creation:** Mathematical models representing mathematical aspects of the world are created
2. **Model exploration:** The models are explored by:
 - a. Stating and proving conjectures
 - b. Performing calculations
3. **Model connection:** The models are connected to each other so that results obtained in one model can be used in other models

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Proof

- Mathematical proof is an essential component of the mathematics process which is unique to mathematics
- It is a method of **justification, communication, and discovery**

- An **informal proof** is a convincing argument that a statement about a model is true

- A **formal proof** is a logical deduction from a set of premises to a conclusion

- A formal proof can be presented in two ways:
 - As a **description** of the actual deduction
 - As a **prescription** for creating the deduction

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Lakatosian View

- Mathematical reasoning is dialectical
 - Dialectic between a theory and its theorems
 - Dialectic between a conjecture and its proof
- New mathematics is discovered by analyzing the proofs of conjectures according to the **method of proof and refutations**
 - The definition-theorem-proof style of presentation hides the true nature of mathematics

Methods for Improving Conjectures

1. Monster barring:
 - Modify some of the definitions so that the counterexample is eliminated
 - Both the conjecture and the proof experiment are unchanged
2. Exception barring:
 - Add a condition to the conjecture so that the counterexample is eliminated
 - Both the conjecture and the proof experiment are changed
3. Lemma incorporation:
 - Make the truth of the subconjecture a condition to the conjecture
 - The conjecture but not the proof experiment is changed

Method of Proofs and Refutations

1. Propose a **conjecture** (which may actually be false)
2. Formulate a **proof experiment** that reduces the conjecture to a set of **subconjectures** (lemmas)

3. Look for **local counterexamples** to the subconjectures

4. If a counterexample is not a **global counterexample** to the conjecture, use it to improve the subconjecture
5. If it is a global counterexample, use it to improve the conjecture
6. Start the process over with the improved conjecture

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Philosophical Questions

- Are mathematical ideas created or discovered?
- What is the nature of mathematical entities?
- Why is mathematics so useful?