

<div data-bbox="1294 371 1321 707" data-label="Text"> <p>CS 773 Winter 2002</p> </div> <div data-bbox="1196 226 1235 848" data-label="Section-Header"> <h2>02. What is Mathematics?</h2> </div> <div data-bbox="1107 380 1130 697" data-label="Text"> <p>Instructor: W. M. Farmer</p> </div> <div data-bbox="1037 388 1060 690" data-label="Text"> <p>Revised : 2 January 2002</p> </div> <div data-bbox="847 846 862 856" data-label="Text"> <p>1</p> </div>	<div data-bbox="1451 1171 1482 1425" data-label="Section-Header"> <h3>Popular View</h3> </div> <div data-bbox="979 1192 1385 1913" data-label="List-Group"> <ul style="list-style-type: none"> • The essence of mathematics is a huge body of concepts and facts about such things as time, measure, pattern, space, and logical consequence • New concepts and facts are discovered by the definition-theorem-proof process • Mathematics is infallible <ul style="list-style-type: none"> – Old concepts and facts are immutable • Conjectures are either proved with a proof or refuted with a counterexample </div> <div data-bbox="847 1850 862 1860" data-label="Text"> <p>3</p> </div>
<div data-bbox="688 170 719 676" data-label="Section-Header"> <h3>Hallmarks of Mathematics</h3> </div> <div data-bbox="134 180 644 909" data-label="List-Group"> <ol style="list-style-type: none"> 1. Abstraction 2. Symbolic methods 3. Conditional reasoning 4. Proof 5. Rigor <ul style="list-style-type: none"> (a) Unambiguous language (b) No hidden assumptions (c) Conclusions follow from assumptions 6. High (and often unexpected) applicability to the real world 7. Extremely long historical development </div> <div data-bbox="84 846 99 856" data-label="Text"> <p>2</p> </div>	<div data-bbox="688 1171 719 1751" data-label="Section-Header"> <h3>Mathematics-as-Process View</h3> </div> <div data-bbox="584 1171 641 1913" data-label="Text"> <p>The essence of mathematics is a process consisting of three intertwined activities:</p> </div> <div data-bbox="196 1182 535 1913" data-label="List-Group"> <ol style="list-style-type: none"> 1. Model creation : Mathematical models representing mathematical aspects of the world are created 2. Model exploration: The models are explored by: <ol style="list-style-type: none"> a. Stating and proving conjectures b. Performing calculations 3. Model connection: The models are connected to each other so that results obtained in one model can be used in other models </div> <div data-bbox="84 1850 99 1860" data-label="Text"> <p>4</p> </div>

Proof

- Mathematical proof is an essential component of the mathematics process which is unique to mathematics
- It is a method of **justification, communication, and discovery**
- An **informal proof** is a convincing argument that a statement about a model is true
- A **formal proof** is a logical deduction from a set of premises to a conclusion
 - As a **description** of the actual deduction
 - As a **prescription** for creating the deduction

5

Method of Proofs and Refutations

1. Propose a **conjecture** (which may actually be false)
2. Formulate a **proof experiment** that reduces the conjecture to a set of **subconjectures** (lemmas)
3. Look for **local counterexamples** to the subconjectures
4. If a counterexample is not a **global counterexample** to the conjecture, use it to improve the subconjecture
5. If it is a global counterexample, use it to improve the conjecture
6. Start the process over with the improved conjecture

7

Lakatosian View

- Mathematical reasoning is dialectical
 - Dialectic between a theory and its theorems
 - Dialectic between a conjecture and its proof
- New mathematics is discovered by analyzing the proofs of conjectures according to the **method of proof and refutations**
- The definition-theorem-proof style of presentation hides the true nature of mathematics

6

Methods for Improving Conjectures

1. Monster barring:
 - Modify some of the definitions so that the counterexample is eliminated
 - Both the conjecture and the proof experiment are unchanged
2. Exception barring:
 - Add a condition to the conjecture so that the counterexample is eliminated
 - Both the conjecture and the proof experiment are changed
3. Lemma incorporation:
 - Make the truth of the subconjecture a condition to the conjecture
 - The conjecture but not the proof experiment is changed

8

Philosophical Questions

- Are mathematical ideas created or discovered?
- What is the nature of mathematical entities?
- Why is mathematics so useful?