CS 773 Winter 2002

02. What is Mathematics?

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Hallmarks of Mathematics

1. Abstraction
2. Symbolic methods
3. Conditional reasoning
4. Proof
5. Rigor
   (a) Unambiguous language
   (b) No hidden assumptions
   (c) Conclusions follow from assumptions
6. High (and often unexpected) applicability to the real world
7. Extremely long historical development

Popular View

- The essence of mathematics is a huge body of concepts and facts about such things as time, measure, pattern, space, and logical consequence
- New concepts and facts are discovered by the definition-theorem-proof process
- Mathematics is infallible
  - Old concepts and facts are immutable
- Conjectures are either proved with a proof or refuted with a counterexample

Mathematics-as-Process View

The essence of mathematics is a process consisting of three intertwined activities:

1. **Model creation**: Mathematical models representing mathematical aspects of the world are created
2. **Model exploration**: The models are explored by:
   a. Stating and proving conjectures
   b. Performing calculations
3. **Model connection**: The models are connected to each other so that results obtained in one model can be used in other models

Proof

- Mathematical proof is an essential component of the mathematics process which is unique to mathematics
- It is a method of **justification**, **communication**, and **discovery**
- An **informal proof** is a convincing argument that a statement about a model is true
- A **formal proof** is a logical deduction from a set of premises to a conclusion
- A formal proof can be presented in two ways:
  - As a **description** of the actual deduction
  - As a **prescription** for creating the deduction

Lakatosian View

- Mathematical reasoning is dialectical
  - Dialectic between a theory and it theorems
  - Dialectic between a conjecture and its proof
- New mathematics is discovered by analyzing the proofs of conjectures according to the **method of proof and refutations**
- The definition-theorem-proof style of presentation hides the true nature of mathematics

Method of Proofs and Refutations

1. Propose a **conjecture** (which may actually be false)
2. Formulate a **proof experiment** that reduces the conjecture to a set of **subconjectures** (lemmas)
3. Look for **local counterexamples** to the subconjectures
4. If a counterexample is not a **global counterexample** to the conjecture, use it to improve the subconjecture
5. If it is a global counterexample, use it to improve the conjecture
6. Start the process over with the improved conjecture

Methods for Improving Conjectures

1. Monster barring:
   - Modify some of the definitions so that the counterexample is eliminated
   - Both the conjecture and the proof experiment are unchanged
2. Exception barring:
   - Add a condition to the conjecture so that the counterexample is eliminated
   - Both the conjecture and the proof experiment are changed
3. Lemma incorporation:
   - Make the truth of the subconjecture a condition to the conjecture
   - The conjecture but not the proof experiment is changed

Philosophical Questions

- Are mathematical ideas created or discovered?
- What is the nature of mathematical entities?
- Why is mathematics so useful?