

History

- Euclid (325–265 BC) used the axiomatic method to present the mathematics known in his time in the **Elements**
 - The axioms were considered truths

08. Practical Application of The Axiomatic Method

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- The development of **noneuclidean geometry** by Bolyai, Gauss, and Lobachevskii (early 1800s) showed that axioms may be considered as just assumptions
- Whitehead and Russell formalized a major portion of mathematics in the **Principia Mathematica** (1910–1913)
- Bourbaki (mid 1900s) used the axiomatic method to codify mathematics in the 30 volume **Éléments de mathématique**
- Several libraries of formalized mathematics have been developed since the late 1980s using interactive theorem provers: HOL, IMPS, Isabelle, Mizar, Nqthm, Nuprl, PVS

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What is the Axiomatic Method?

1. A mathematical model is expressed as a set of axioms (in a language) called an **axiomatic theory**
2. New concepts are introduced by making **definitions**
3. Assertions about the model are stated as **theorems** and proved from the axioms

Notes:

- The axiomatic method is a method of **presentation**, not a method of **discovery** (Lakatos)
- The axiomatic method can be used as a method of **organization**

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Example: Theory of Partial Order

- Language: A first-order logic language with a 2-place predicate symbol \leq
 - $a \leq b$ is intended to mean a is less than or equal to b
- Axioms:
 - **Reflexivity.** $\forall x. x \leq x$
 - **Transitivity.** $\forall x, y, z. (x \leq y \wedge y \leq z) \supset x \leq z$
 - **Antisymmetry.** $\forall x, y. (x \leq y \wedge y \leq x) \supset x = y$
- The theory has infinitely many nonisomorphic models

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Example: Peano Arithmetic

- Language: A second-order logic language with a constant symbol 0 and 1-place function symbol S
 - 0 is intended to represent the number zero
 - S is intended to represent the successor function, i.e., $S(a)$ means $a + 1$
- Axioms:
 - **0 has no predecessor.** $\forall x. \neg(0 = S(x))$
 - **S is injective.** $\forall x, y. S(x) = S(y) \supset x = y$
 - **Induction principle.**
 $\forall P. (P(0) \wedge \forall x. P(x) \supset P(S(x))) \supset \forall x. P(x)$
 - Second-order Peano arithmetic is **categorical**, i.e., it has exactly one model up to isomorphism

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Benefits of Axiomatic Theories

- Conceptual clarity: inessential details are omitted
- Generality: theorems hold in all models
- Dual purpose: a theory can be viewed as:
 - An abstract mathematical model
 - A specification of a collection of mathematical models
- Interpretations are information conduits!

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Example: Theory of Computer Networks

- Theory name: Networks
- Language: Many-sorted first-order logic language with the following sorts and function symbols:

Sorts	Function symbols
boxes	box-of-interface
wires	wire-of-interface
interfaces	address-of-interface
addresses	

- Example axioms:
 - “Every box has a unique loopback interface”
 - “The address of a loopback interface is 127.0.0.1”

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Example: Theory of Bipartite Graphs

- Theory name: Bipartite Graphs
- Language: Many-sorted first-order logic language with the following sorts and function symbols:

Sorts	Operators
red-nodes	red-node-of-edge
blue-nodes	blue-node-of-edge
edges	

- No explicit axioms

Example: Bipartite Graphs to Networks

- Let $\Phi_{BG \rightarrow N}$ be the translation from Bipartite Graphs to Networks defined by:
 - red-nodes \mapsto boxes
 - blue-nodes \mapsto wires
 - edges \mapsto interfaces
- $\Phi_{BG \rightarrow N}$ has no obligations
- $\Phi_{BG \rightarrow N}$ is an interpretation
 - “Transitivity of red-to-red connectivity” maps to “transitivity of box-to-box connectivity”
- $\Phi_{BG \rightarrow BG}$ has no obligations:
 - “Transitivity of red to red connectivity” maps to “transitivity of blue to blue connectivity”
- $\Phi_{BG \rightarrow BG}$ is an interpretation

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Example: Symmetry Interpretation

- Let $\Phi_{BG \rightarrow BG}$ be the translation from Bipartite Graphs to Bipartite Graphs defined by:
 - red-nodes \mapsto blue-nodes
 - blue-nodes \mapsto red-nodes
 - edges \mapsto edges
- $\Phi_{BG \rightarrow BG}$ has no obligations:
 - “Transitivity of red to red connectivity” maps to “transitivity of blue to blue connectivity”

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Two Versions of the Axiomatic Method

Formalized Mathematics

- 1. **Big Theory:** A body of mathematics is entirely represented in one theory
 - Often a powerful, highly expressive theory like set theory is selected
 - All reasoning is performed within this single theory
- 2. **Little Theories:** A body of mathematics is represented as a network of theories
 - Bigger theories are composed of smaller theories
 - Theories are linked by interpretations
 - Reasoning is distributed over the network
 - * Theory creation
 - * Conservative theory extension
 - * Theory exploration
 - * Theory interpretation

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Benefits of Little Theories

- Mathematics can be developed using the most appropriate vocabulary at the most appropriate level of abstraction
- Emphasizes reuse: if A is a theorem of T , then A may be reused in any "instance" of T
- Enables perspective switching
 - Enables parallel development
- Inconsistency can be isolated: there are no interpretations of an inconsistent theory in a consistent theory, so inconsistency cannot spread from one theory to another

Theory Creation

- Theories can be created in a several ways:
 - From scratch
 - By forming a union of a set of theories
 - By adding new vocabulary and axioms to a theory
 - By instantiating a parameterized theory
 - By instantiating a theory via an interpretation
- A theory may be required to contain a **kernel theory** which includes the machinery common to all theories

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Conservative Theory Extension

- A conservative extension T' of T adds new machinery to T without compromising the original machinery of T
- The **obligation** of a purported conservative extension is a formula that implies that the extension is conservative
- Since T and T' are essentially the same theory, T' can be implemented by overwriting T
 - Avoids a proliferation of closely related theories
- There are two important kinds of conservative extensions that add new vocabulary to a theory:
 - Definitions
 - Profiles

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Definitions

- A **definition** is a conservative extension that adds a new symbol s and a defining axiom $A(s)$ to a theory T
 - In some logics, the defining axiom can have the form $s = D$ (where s does not occur in D)
- The obligation of the definition is
$$\exists !x. A(x)$$
- The symbol s can usually be eliminated from any new expression of involving s

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Profiles

- A **profile** is a conservative extension that adds a set $\{s_1, \dots, s_n\}$ of symbols and a profiling axiom $A(s_1, \dots, s_n)$ to a theory T
- The obligation of the profile is
$$\exists x_1, \dots, x_n. A(x_1, \dots, x_n)$$
- The symbols s_1, \dots, s_n cannot usually be eliminated from expressions involving s_1, \dots, s_n
- Profiles can be used for introducing:
 - Underspecified objects
 - Recursively defined functions
 - Abstract datatypes

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Theory Exploration

- The logical consequences of a theory are explored by:
 - Proving conjectures
 - Performing computations
- Products of theory exploration:
 - Theorems
 - Proofs
 - Counterexamples
 - Computations
- Tools of theory exploration:
 - Theorems
 - Transformers

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Theorems

- Facts about a theory are recorded as theorems
- A theorem is usually installed in a theory only if it has been verified by a proof
 - A theorem verified by a decision procedure
 - A theorem verified by a counterexample
 - A theorem imported via an interpretation
 - A theorem shown by a metatheorem
- A theorem may sometimes be installed without a proof:
 - Show relative consistency of theories
 - Show theory extension conservativity

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Transformers

- A **transformer** is a function that maps the expressions of a language L to the expressions of a language L'
 - Usually, $L \leq L'$, $L' \leq L$, or $L = L'$
- A transformer can be used to represent an expression transforming operation such as an evaluator, a simplifier, a rewrite rule, a rule of inference, a decision procedure, or an interpretation of one language in another
- Sound transformers can be:
 - Generated from theorems (e.g., theorem macetes)
 - Constructed from other transformers using certain constructors (e.g., compound macetes)
 - Obtained by instantiating abstract transformers (e.g., algebraic and order processors)
 - Manually defined and verified

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Interpretations

- Theory interpretations can be used to:
 - Transport theorems, definitions, and profiles
 - Instantiate theories
 - Compare the strength of theories
 - Show relative consistency of theories
 - Show theory extension conservativity
- Logic interpretations can be used to interpret a theory in one logic in a theory of another logic

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