SE 2F03 Fall 2005
Assignment 4 Solutions

Instructor: William M. Farmer

Revised: 14 December 2005

(1) [30 pts.] Using lambda-notation, definite description, if (an if-then-else term constructor), and types, write expressions that denote the following functions and predicates:

(a) The identity function on $\mathbf{R}$, the real numbers.

Answer: $\lambda\, x : \mathbf{R} \,.\, x$.

(b) The empty function on $\mathbf{Q}$, the rational numbers.

Answer: $\lambda\, x : \mathbf{Q} \,.\, 1/0$ or $\lambda\, x : \mathbf{Q} \,.\, \bot_{\mathbf{Q}}$

(c) The curryed form of $+ : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$.

Answer: $\lambda\, x : \mathbf{R} \,.\, \lambda\, y : \mathbf{R} \,.\, x + y$.

(d) The predicate that maps the rational numbers to $\textsc{t}$ (true) and the irrational numbers to $\textsc{f}$ (false).

Answer: $\lambda\, x : \mathbf{R} \,.\, \mathsf{if}(\mathsf{rational}(x), \mathsf{T}, \mathsf{F})$ or $\lambda\, x : \mathbf{R} \,.\, \mathsf{rational}(x)$.

(e) The function that maps each rational number q to the largest integer i such that $i \leq q$ (i.e., to the "floor" of q).

Answer: $\lambda\, q : \mathbf{Q} \,.\, \mathrm{I}\, i : \mathbf{Z} \,.\, i \leq q \wedge \neg \exists\, j : \mathbf{Z} \,.\, i < j \wedge j \leq q$.

(f) The restriction of the multiplication function $* : \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$ to nonnegative integers.

Answer: $\lambda\, x, y : \mathbf{Z} \,.\, \mathsf{if}(0 \leq x \wedge 0 \leq y, x * y, \bot_{\mathbf{Z}})$.

(g) The function that maps each real number to its cubic root.

Answer: $\lambda\, x : \mathbf{R} \,.\, \mathrm{I}\, y : \mathbf{R} \,.\, y * y * y = x$.

(h) The predicate that, given a set of real numbers represented as a predicate $p : \mathbf{R} \rightarrow *$, returns as input $\textsc{t}$ if the set is nonempty and $\textsc{f}$ otherwise.

Answer: $\lambda\, p : (\mathbf{R} \rightarrow *) \,.\, \exists\, x : \mathbf{R} \,.\, p(x)$.

(i) The function that, given two functions $f, g : \mathbf{Z} \rightarrow \mathbf{Z}$ as input, returns as output the function that maps each integer x to $f(x) + g(x)$.

Answer: $\lambda\, f, g : (\mathbf{Z} \rightarrow \mathbf{Z}) \,.\, \lambda\, x : \mathbf{Z} \,.\, f(x) + g(x)$.

(j) The function that, given two functions $f, g : \mathbf{Q} \to \mathbf{Q}$ as input, returns as output the composition of f and g.
Answer: $\lambda\, f, g : (\mathbf{Q} \to \mathbf{Q})\ .\ \lambda\, x : \mathbf{Q}\ .\ f(g(x))$.

(2) [10 pts.] Reduce the following functions applications using only beta-reduction:

(a) $(\lambda\, y : \mathbf{R}\ .\ (\lambda\, x\ .\ f(x))(y))$.
Answer: $\lambda\, x\ .\ f(x)$.

(b) $(\lambda\, x : \mathbf{Z}\ .\ (\lambda\, x\ .\ x^2)(2) + x)(7)$.
Answer: $2^2 + 7$.

(c) $(\lambda\, u : \mathbf{Q}\ .\ \mathsf{if}(u = 0, u, -u))(g(f(17) + f(g(2, 31))))$
Answer: $\mathsf{if}(g(f(17) + f(g(2, 31))) = 0,$
$g(f(17) + f(g(2, 31))),$
$-\, g(f(17) + f(g(2, 31))))$.

(d) $(\lambda\, f : (\mathbf{R} \to (\mathbf{R} \to \mathbf{R}))\ .\ (\lambda\, x : \mathbf{R}\ .\ (\lambda\, y : \mathbf{R}\ .\ f(x)(y))))(+)(2)(3)$.
Answer: $+(2)(3)$.

(e) $(\lambda\, f : (\mathbf{R} \to (\mathbf{R} \to \mathbf{R}))\ .\ f(f(2)))(\lambda\, x : \mathbf{R}\ .\ x^x)$.
Answer: $(2^2)^{(2^2)}$.

(3) [10 pts.] Which of the following expressions are undefined? Explain your answer in each case.

(a) $(\lambda\, x : \mathbf{R}\ .\ \sqrt{x})$.
Answer: Defined because it denotes the square root function.

(b) $(\lambda\, x : \mathbf{R}\ .\ \sqrt{x})(-5)$.
Answer: Undefined because $\sqrt{-5}$ is undefined.

(c) $(\iota\, z : \mathbf{R}\ .\ 0 = 0 * z)$.
Answer: Undefined because every real number satisfies the body of the definite description.

(d) $(\epsilon\, z : \mathbf{R}\ .\ 0 = 0 * z)$.
Answer: Defined because every, and hence at least one, real number satisfies the body of the indefinite description.

(e) $1/0 = 1/0$.
Answer: Defined because formulas are always defined. This formula would be true in a logic in which all expressions are defined, but would probably be false in a logic in which expressions may be undefined.

(4) [10 pts.] What is the type of each of the following expressions of STT?

(a) $(\lambda\, x : \iota \,.\, \lambda\, y : \iota \,.\, f(x)(y))$.
Answer: $(\iota \rightarrow (\iota \rightarrow \iota))$ assuming the type of f is $(\iota \rightarrow (\iota \rightarrow \iota))$.

(b) $(\lambda\, x : \iota \,.\, \lambda\, f : (\iota \rightarrow \iota) \,.\, f(x))$.
Answer: $(\iota \rightarrow ((\iota \rightarrow \iota) \rightarrow \iota))$.

(c) $(\lambda\, x : \iota \,.\, \lambda\, f : (\iota \rightarrow \iota) \,.\, f(x) = a)$.
Answer: $(\iota \rightarrow ((\iota \rightarrow \iota) \rightarrow *))$.

(d) $(\lambda\, x : \iota \,.\, x = x)(b)$.
Answer: $*$.

(e) $(\mathrm{I}\, z : \iota \,.\, z \neq z)$.
Answer: ι.

(5) [30 pts.] Translate the following statements into STT:

(a) A unary function f of type $(\iota \rightarrow \iota)$ is injective.
Answer: $\forall\, x, y : \iota \,.\, f(x) = f(y) \Rightarrow x = y$.

(b) A binary predicate p of type $(\iota \rightarrow (\iota \rightarrow *))$ is a transitive relation.
Answer: $\forall\, x, y, z : \iota \,.\, (p(x)(y) \wedge p(y)(z)) \Rightarrow p(x)(z)$.

(c) A unary predicate s of type $(\iota \rightarrow *)$ represents a singleton set.
Answer: $\exists\, x : \iota \,.\, (s(x) \wedge (\forall\, y : \iota \,.\, s(y) \Rightarrow y = x))$.

(d) There exists a function f from individual to individuals such that, for every individual x, $f(f(x)) = x$.
Answer: $\exists\, f : (\iota \rightarrow \iota) \,.\, \forall\, x : \iota \,.\, f(f(x)) = x$.

(e) Two functions from individuals to individuals are equal iff they map the individuals in exactly the same way.
Answer: $\forall\, f, g : (\iota \rightarrow \iota) \,.\, f = g \Leftrightarrow \forall\, x : \iota \,.\, f(x) = g(x)$.

(f) Two sets of individuals are equal iff they have exactly the same members. (Represent the sets as unary predicates.)
Answer: $\forall\, s, t : (\iota \rightarrow *) \,.\, s = t \Leftrightarrow \forall\, x : \iota \,.\, s(x) \Leftrightarrow t(x)$.

(g) Two individuals are the equal iff they satisfy exactly the same set of properties.
Answer: $\forall\, a, b : \iota \,.\, a = b \Leftrightarrow \forall\, p : (\iota \rightarrow *) \,.\, p(a) \Leftrightarrow p(b)$.

(h) There exists a unique individual that satisfies a unary predicate q.
Answer: $\exists\, x : \iota \,.\, (q(x) \wedge (\forall\, y : \iota \,.\, q(y) \Rightarrow y = x))$.

(i) The unique individual that satisfies a unary predicate q equals the individual represented by the constant a.

Answer: $(\mathrm{I}\, x : \iota \,.\, q(x)) = a$.

(j) For every function f of type $(\iota \to \iota)$, there exists a predicate p of type $(\iota \to (\iota \to *))$ such that p is the graph of f.

Answer: $\forall\, f : (\iota \to \iota) \,.\, \exists p : (\iota \to (\iota \to *)) \,.\, \forall\, x, y : \iota \,.\, p(x)(y) \Leftrightarrow f(x) = y$.

(6) [5 pts.] Let $L = (\{f\}, \tau)$, where $\tau(f) = (\iota \to \iota)$, be a language of STT. Find a set Γ of sentences of L such that M is a model of $T = (L, \Gamma)$ iff the interpretation of f in M is a bijection, i.e., a total function that is injective (one-to-one) and surjective (onto).

Answer: Γ is the set of the following sentences of L:

- $\forall\, x, y : \iota \,.\, f(x) = f(y) \Rightarrow x = y$ (injectivity).
- $\forall\, y : \iota \,.\, \exists\, x : \iota \,.\, f(x) = y$ (surjectivity).

(7) [5 pts.] Find a language L of STT and a set Γ of sentences of L such that M is a model of $T = (L, \Gamma)$ iff M is a Boolean algebra.

Answer: Let $L = (\{0, 1, {}^{-}, +, \cdot\}, \tau)$ be the language of STT such that:

- $\tau(0) = \tau(1) = \iota$.
- $\tau({}^{-}) = (\iota \to \iota)$.
- $\tau(+) = \tau(\cdot) = (\iota \to (\iota \to \iota))$.

and let Γ be the set of the following sentences of L:

- $\forall\, x, y, z : \iota \,.\, +(+(x)(y))(z) = +(x)(+(y)(z))$.
- $\forall\, x, y, z : \iota \,.\, \cdot(\cdot(x)(y))(z) = \cdot(x)(\cdot(y)(z))$.
- $\forall\, x, y : \iota \,.\, +(x)(y) = +(y)(x)$.
- $\forall\, x, y : \iota \,.\, + \cdot (x)(y) = \cdot(y)(x)$.
- $\forall\, x, y, z : \iota \,.\, +(x)(\cdot(y)(z)) = \cdot(+(x)(y))(+(x)(z))$.
- $\forall\, x, y, z : \iota \,.\, \cdot(x)(+(y)(z)) = +(\cdot(x)(y))(\cdot(x)(z))$.
- $\forall\, x : \iota \,.\, +(x)(0) = x$.
- $\forall\, x : \iota \,.\, \cdot(x)(1) = x$.
- $\forall\, x : \iota \,.\, +(x)(\overline{x}) = 1$.

- $\forall\, x : \iota \,.\, \cdot(x)(\overline{x}) = 0$.

Notice that we have written the axioms using entirely prefix notation; the axioms would be more readable if infix notation were used for $+$ and $\cdot$ instead.

$T = (L, \Gamma)$ is an STT theory of Boolean algebras.