SE 2F03 Fall 2005

Final Examination Answer Key

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(1) [2 pts.] The IMPS logic is a version of first-order logic. Is this statement true or false?

A.) True.

B.) **[False.]**

(2) [2 pts.] The formula $\neg p$ is logically equivalent to $p \mid p$. Is this statement true or false?

A.) **[True.]**

B.) False.

(3) [2 pts.] In STT, the type ι represents the type of individuals. Is this statement true or false?

A.) **[True.]**

B.) False.

(4) [2 pts.] Every valid formula is satisfiable. Is this statement true or false?

A.) **[True.]**

B.) False.

(5) [2 pts.] Everything that can be formalized in propositional logic can be also formalized in first-order logic. Is this statement true or false?

A.) **[True.]**

B.) False.

(6) [2 pts.] A proof system is useless if it is not both sound and complete. Is this statement true or false?

A.) True.

B.) **[False.]**

(7) [2 pts.] $T \models A$ is false if T has no models. Is this statement true or false?

A.) True.

B.) $\boxed{\text{False.}}$

(8) [2 pts.] Simple type theory is a kind of higher-order logic? Is this statement true or false?

A.) $\boxed{\text{True.}}$

B.) False.

(9) [2 pts.] Who is generally considered the greatest logician of all time?

A.) Aristotle.

B.) Goethe.

C.) $\boxed{\text{Gödel.}}$

D.) Boole.

(10) [2 pts.] Which of the following definite descriptions is improper?

A.) $\mathrm{I}\, x : \mathbf{R} \,.\, x = 2.$

B.) $\mathrm{I}\, x : \mathbf{R} \,.\, x = \sqrt{2}.$

C.) $\boxed{\mathrm{I}\, x : \mathbf{R} \,.\, x + x = x * x.}$

D.) $\mathrm{I}\, x : \mathbf{R} \,.\, 1 + x + x = x * x.$

(11) [2 pts.] Let $L = (\{0\}, \{+\}, \{=\})$ be a language of FOL where $+$ is binary, and suppose $T = (L, \Gamma)$ is a theory of monoids. Which of the following formulas is not a logical consequence of T?

A.) $\forall\, x, y, z \,.\, ((x + y) + z) = (x + (y + z)).$

B.) $\boxed{\forall\, x, y \,.\, (x + y) = (y + x).}$

C.) $\forall\, x \,.\, x + 0 = x.$

D.) $0 = 0 + 0.$

(12) [2 pts.] It is possible to write a computer program that can decide the validity of any formula of

A.) $\boxed{\text{PROP.}}$

B.) FOL.

C.) STT.

D.) All of the above.

(13) [2 pts.] The dual of $\Rightarrow$ (implication) is

A.) $\vee$.

B.) $\wedge$.

C.) $\Leftrightarrow$.

D.) $\boxed{\text{None of the above.}}$

(14) [2 pts.] $(\lambda\, x : \mathbf{Z} \,.\, (\lambda\, x : \mathbf{Z} \,.\, f(x)))(x+1)$ beta-reduces to

A.) $\lambda\, x : \mathbf{Z} \,.\, (\lambda\, x : \mathbf{Z} \,.\, f(x))$.

B.) $\boxed{\lambda\, x : \mathbf{Z} \,.\, f(x).}$

C.) $\lambda\, x : \mathbf{Z} \,.\, f(x+1)$.

D.) $f(x+1)$.

(15) [2 pts.] What is the value of the expression

$$\iota\, f : \mathbf{N} \to \mathbf{N} \,.\, \forall\, x : \mathbf{N} \,.\, f(x) = \mathsf{if}(x = 0, 1, x * f(x-1))?$$

A.) True.

B.) $f(x)$.

C.) $x!$.

D.) $\boxed{\text{The factorial function.}}$

(16) [2 pts.] Which of the following LTL formulas is always true?

A.) $\varphi \mathrm{~U~} \psi \to \mathrm{F}\ \varphi$.

B.) $\boxed{\psi \mathrm{~U~} \varphi \to \mathrm{F}\ \varphi.}$

C.) $\varphi \mathrm{~W~} \psi \to \mathrm{F}\ \varphi$.

D.) $\psi \mathrm{~W~} \varphi \to \mathrm{F}\ \varphi$.

(17) [2 pts.] Suppose $\oplus$ is the curryed form of the addition function $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$. Then $\oplus(2)$ equals

A.) $\boxed{\lambda\, x : \mathbf{R} \,.\, 2 + x.}$

B.) $\forall\, x : \mathbf{R} \,.\, 2 + x$.

C.) $\iota\, x : \mathbf{R} \,.\, 2 + x$.

D.) $2 + 2$.

(18) [2 pts.] Which of the following natural deduction rules discharges assumptions?

A.) $\vee$ elimination.

B.) $\exists x$ elimination.

C.) $\rightarrow$ introduction.

D.) $\boxed{\text{All of the above.}}$

(19) [2 pts.] Modus ponens is the same rule as

A.) $\rightarrow$ introduction.

B.) $\boxed{\rightarrow \text{ elimination.}}$

C.) Proof by contradiction.

D.) The law of excluded middle.

(20) [2 pts.] Let $L = (\{0\}, \{+\}, \{=\})$ be a language of FOL where $+$ is binary. Which of the following is a statement in the metalanguage of L?

A.) $\forall x, y \,.\, x + y = y + x.$

B.) $0 = 0.$

C.) $x + 0 = x.$

D.) $\boxed{\models \forall x \,.\, x + 0 = x.}$

(21) [5 pts.] Give an example of a domain D and a binary predicate p on D such that

$$\forall x \in D \,.\, \exists y \in D \,.\, p(x, y)$$

is true, but

$$\exists y \in D \,.\, \forall x \in D \,.\, p(x, y)$$

is false.

Answer: Let D be the set of all human beings, and let $p : D \times D$ be the predicate such $p(x, y)$ holds iff x is a child of y.

(22) Let A be the formula

$$(p \wedge (q \to \neg r)) \to (r \to (\neg p \vee q))$$

of propositional logic.

A.) [5 pts.] Compute the truth table of A.

Answer:

p	q	r	$(p \wedge (q \to \neg r)) \to (r \to (\neg p \vee q))$							
T	T	T	F	F	F	T	T	F	T	T
T	T	F	T	T	T	T	T	F	T	T
T	F	T	T	T	F	F	F	F	F	F
T	F	F	T	T	T	T	T	F	F	T
F	T	T	F	F	F	T	T	T	T	T
F	T	F	F	T	T	T	T	T	T	T
F	F	T	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T	T	T

B.) [5 pts.] Construct a formula C of propositional logic in conjunctive normal form such that A and C are logically equivalent.

Answer: $\neg p \vee q \vee \neg r$.

C.) [5 pts.] Construct a formula D of propositional logic in disjunctive normal form such that A and D are logically equivalent.

Answer: $(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$.

(23) Let $L = \{p, q\}$ be a language of LTL and consider the model $M = (S, \to, I)$ of L such that $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, $s_1 \to s_2$, $s_2 \to s_3$, $s_3 \to s_4$, $s_4 \to s_5$, $s_5 \to s_6$, $s_6 \to s_1$, $s_2 \to s_4$, $s_2 \to s_5$, $s_4 \to s_4$, $s_5 \to s_2$, $I(s_1, p) = \text{F}$, $I(s_1, q) = \text{F}$, $I(s_2, p) = \text{T}$, $I(s_2, q) = \text{F}$, $I(s_3, p) = \text{T}$, $I(s_3, q) = \text{T}$, $I(s_4, p) = \text{F}$, $I(s_4, q) = \text{T}$, $I(s_5, p) = \text{T}$, $I(s_5, q) = \text{T}$, $I(s_6, p) = \text{T}$, and $I(s_6, q) = \text{F}$. (The figure is not shown.)

A.) [5 pts.] Find a path π starting at s_1 such that

$$\pi \models \text{G}\,(p \to \neg q).$$

Answer: $\pi = s_1 \to s_2 \to s_4 \to s_4 \to \cdots$.

B.) [5 pts.] Find the set $S' \subseteq S$ of states such that $s \in S'$ iff

$$M, s \models \neg p \mathrm{~W~} (p \wedge q).$$

Answer: $S' = \{s_3, s_4, s_5\}$.

(24) [5 pts.] Let $T = (L, \Gamma)$ be a theory of FOL such that $L = (\emptyset, \{g\}, \{=, r\})$, where g is unary and r is binary, and Γ contains the following sentences:

$\forall\, x, y \,.\, r(x, y) \Rightarrow r(y, x)$.
$\forall\, x \,.\, r(x, g(x))$.

Translate T into a theory $T' = (L', \Gamma')$ of STT by stating what L' and Γ' are.

Answer:

$L' = (\{g, r\}, \tau)$ where $\tau(g) = (\iota \to \iota)$ and $\tau(r) = (\iota \to (\iota \to *))$.
$\Gamma' = \{\forall\, x, y : \iota \,.\, r(x)(y) \Rightarrow r(y)(x),\ \forall\, x : \iota \,.\, r(x)(g(x))\}$.

(25) [5 pts.] Let $L = (\{a\}, \{f\}, \{=\})$, where f is unary, be a language of FOL. Find a set Γ of sentences of L such that $M = (D, I)$ is a model of $T = (L, \Gamma)$ iff the $I(f)$ is a function whose output value is $I(a)$ for all input values.

Answer: $\Gamma = \{\forall\, x \,.\, f(x) = a\}$.

(26) [10 pts.] Prove the sequent

$$\forall\, x \,.\, f(f(x)) = x \;\vdash\; (\forall\, x \,.\, P(f(x))) \to P(a)$$

by natural deduction.

Proof:

1	$\forall\, x \,.\, f(f(x)) = x$	premise
2	$\forall\, x \,.\, P(f(x))$	assumption
3	$P(f(f(a)))$	$\forall x$ e 2
4	$f(f(a)) = a$	$\forall x$ e 1
5	$P(a)$	=e 4,3
6	$(\forall\, x \,.\, P(f(x))) \to P(a)$	$\to$i 2–5

(27) [10 pts.] Prove the sequent

$$\forall x \,.\, \neg B(x),\ \exists y \,.\, B(y) \vee R(y)\ \vdash \exists z \,.\, R(z)$$

by natural deduction.

Proof:

1		$\forall x \,.\, \neg B(x)$	premise
2		$\exists y \,.\, B(y) \vee R(y)$	premise
3	y_0	$B(y_0) \vee R(y_0)$	assumption
4		$B(y_0)$	assumption
5		$\neg B(y_0)$	$\forall x$ e 1
6		$\bot$	$\neg$e 4,5
7		$\exists z \,.\, R(z)$	$\bot$e 6
8		$R(y_0)$	assumption
9		$\exists z \,.\, R(z)$	$\exists z$ i 8
10		$\exists z \,.\, R(z)$	$\vee$e 3,4–7,8–9
11		$\exists z \,.\, R(z)$	$\exists y$ e 2, 3–10