Name ______________________________

Student number ______________________________

SE 2F03 Fall 2005

Midterm Test 2 Answer Key

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Revised: 17 November 2005

You have 50 minutes to complete this test consisting of 6 pages and 15 questions. You may use your notes and textbooks. Circle the *best* answer for the multiple choice questions, and write the answer to the other questions in the space provided. Good luck!

(1) [4 pts.] The temporal connective X can be defined using F and R. Is this statement true or false?

(a) True.

(b) **[False.]**

(2) [4 pts.] If a term t is free for a variable x in a formula A, then all occurrences of x in A are free. Is this statement true or false?

(a) True.

(b) **[False.]**

(3) [4 pts.] An unwinding of a transition system is infinite even if the system contains only one state. Is this statement true or false?

(a) **[True.]**

(b) False.

(4) [4 pts.] By the rule of universal instantiation, $T \models \forall x \,.\, \exists y \,.\, q(x, y)$ implies $T \models \exists y \,.\, q(y, y)$. Is this statement true or false?

(a) True.

(b) **[False.]**

(5) [4 pts.] $M = (S, \cup, \cap, \overline{}, \emptyset, \mathbf{N}, =)$, where $\mathbf{N}$ is the set of natural numbers and S is the set of finite subsets of $\mathbf{N}$, is a Boolean algebra. Is this statement true or false?

(a) True.

(b) **[False.]**

(6) [4 pts.] Let A be $\exists x \,.\, (y = x \wedge \forall z \,.\, p(f(x), y))$. Then $A[g(z)/y]$ is

(a) $\exists x \,.\, (y = x \wedge \forall z \,.\, p(f(x), y))$.

(b) $\exists x \,.\, (g(z) = x \wedge \forall z \,.\, p(f(x), y))$.

(c) $\boxed{\exists x \,.\, (g(z) = x \wedge \forall z \,.\, p(f(x), g(z))).}$

(d) $\exists x \,.\, (y = x \wedge \forall z \,.\, p(f(x), g(z)))$.

(7) [4 pts.] If time is represented in a temporal logic by the integers, then the temporal logic would be

(a) Continuous.

(b) $\boxed{\text{Linear and discrete.}}$

(c) Branching and discrete.

(d) Linear and continuous.

(8) [4 pts.] The dual of X is

(a) F.

(b) G.

(c) U.

(d) $\boxed{\text{X.}}$

(9) [4 pts.] The absolute value of a real number r, written $|r|$, would be most directly formalized in first-order logic as

(a) An individual constant.

(b) $\boxed{\text{The application of a function symbol.}}$

(c) The application of a predicate symbol.

(d) An implication.

(10) [4 pts.] $\varphi \,\mathrm{W}\, \bot$ is equivalent to

(a) $\mathrm{F}\,\varphi$.

(b) $\boxed{\mathrm{G}\,\varphi.}$

(c) $\varphi \,\mathrm{U}\, \top$.

(d) $\boxed{\bot \,\mathrm{R}\, \varphi.}$

(11) Let $L = \{p, q\}$ be a language of LTL and consider the model $M = (S, \rightarrow, I)$ of L such that $S = \{s_1, s_2, s_3, s_4\}$, $s_1 \rightarrow s_2$, $s_2 \rightarrow s_2$, $s_3 \rightarrow s_2$, $s_1 \rightarrow s_3$, $s_3 \rightarrow s_4$, $s_4 \rightarrow s_3$, $s_4 \rightarrow s_1$, $I(s_1, p) = \text{T}$, $I(s_1, q) = \text{F}$, $I(s_2, p) = \text{F}$, $I(s_2, q) = \text{T}$, $I(s_3, p) = \text{T}$, $I(s_3, q) = \text{T}$, $I(s_4, p) = \text{F}$, and $I(s_4, q) = \text{F}$. (The figure is not shown.)

(a) [8 pts.] Find the set $S' \subseteq S$ such that $s \in S'$ iff

$$M, s \models \text{G}\,((p \rightarrow \text{X}\,q) \wedge (q \rightarrow \text{X}\,q)).$$

Answer: $S' = \{s_2\}$.

(b) [8 pts.] Find the set Π of paths in M such that $\pi \in \Pi$ iff

$$\pi \models (p \text{ U } \neg q) \wedge \text{G}\,(\neg q \rightarrow \text{X}\,q).$$

Answer: Π is the set of all paths of the form $s_1 \rightarrow \cdots$, $s_4 \rightarrow \cdots$, or $s_3 \rightarrow s_4 \rightarrow \cdots$ that do not contain the transition $s_4 \rightarrow s_1$. For example, Π contains the path $s_3 \rightarrow s_4 \rightarrow s_3 \rightarrow s_2 \rightarrow s_2 \rightarrow \ldots$.

(12) [10 pts.] Express the statement "at some point in the future the atom p is true three times in a row" as an LTL formula.

Answer: $\text{F}\,(p \wedge \text{X}\,p \wedge \text{XX}\,p)$.

(13) [10 pts.] Express the statement "the event r occurs if and only if the event s occurs first" as an LTL formula.

Answer: $(\neg r \text{ W } s) \wedge \text{F}\,(s \rightarrow \text{XF}\,r)$.

(14) [12 pts.] Prove the sequent

$$\forall x \,.\, (p(g(x)) \rightarrow q(x)),\ p(g(f(f(a)))) \vdash \exists y \,.\, q(f(y))$$

by natural deduction.

Proof:

1	$\forall x \,.\, (p(g(x)) \rightarrow q(x))$	premise
2	$p(g(f(f(a))))$	premise
3	$p(g(f(f(a)))) \rightarrow q(f(f(a)))$	$\forall x$ e 1
4	$q(f(f(a)))$	$\rightarrow$e 2,3
5	$\exists y \,.\, q(f(y))$	$\exists y$ i 4

(15) [12 pts.] Prove the sequent

$$\forall x\,.\,(\neg(a = x) \to p(x)),\ \forall x\,.\,(a = x \lor \neg(a = x)),\ p(a) \vdash \forall y\,.\,p(y)$$

by natural deduction. (Hint: Do a case split.)

Proof:

1		$\forall x\,.\,(\neg(a = x) \to p(x))$	premise
2		$\forall x\,.\,(a = x \lor \neg(a = x))$	premise
3		$p(a)$	premise
4	y_0	$a = y_0 \lor \neg(a = y_0)$	$\forall x$ e 2
5		$a = y_0$	assumption
6		$p(y_0)$	=e 5,3
7		$\neg(a = y_0)$	assumption
8		$\neg(a = y_0) \to p(y_0)$	$\forall x$ e 1
9		$p(y_0)$	$\to$e 7,8
10		$p(y_0)$	$\lor$e 4,5–6,7–9
11		$\forall y\,.\,p(y)$	$\forall y$ i 4–10